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ISOTOPIC DEFORMATIONS OF GEODESIC COMPLEXES ON THE 2-SPHERE AND ON THE PLANE[1]

BY STEWART S. CAIRNS

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1. The deformation theorem

Two simplicial complexes, K_0 and K_1, are called *isomorphic* if their respective sets of vertices can be so numbered, P_i and Q_i $(i = 1, 2, \cdots)$, that $Q_{i_0} \cdots Q_{i_m}$ is a cell of K_1 when and only when $P_{i_0} \cdots P_{i_m}$ is a cell of K_0. We will then say that the vertices are *similarly numbered.*

A complex on a euclidean 2-sphere will be referred to as *geodesic* if each of its 1-cells is an arc of great circle shorter than a semi-circle and each of its 2-cells is a spherical triangle less in area than a hemisphere. Every complex throughout this paper will be finite and will be non-singular, in the sense that no two different cells of any one complex will have a point in common.

DEFORMATION THEOREM.[2] *Let K_0 and K_1 be a pair of isomorphic geodesic complexes on a euclidean 2-sphere, S. Let P_i and Q_i $(i = 1, \cdots, n)$ be the vertices, similarly numbered, of K_0 and K_1 respectively. If and only if the isomorphism*[3] *between K_0 and K_1 can be extended into an orientation-preserving self-homeomorphism of S, it is possible to define, for every t $(0 \leqq t \leqq 1)$, a geodesic complex, K_t, with vertices $P_i(t)$ $(i = 1, \cdots, n)$ in such a way that (1) $P_i(0) = P_i$ and $P_i(1) = Q_i$ $(i = 1, \cdots, n)$ (2) K_t and K_0 are isomorphic with vertices similarly numbered (3) as t increases from 0 to 1, $P_i(t)$ traces a continuous curve on S $(i = 1, \cdots, n)$.*

The conclusion of this theorem asserts the existence of an isotopic deformation of K_0 into K_1 induced by a motion of the vertices, all cells remaining geodesic and non-degenerate throughout the motion. The proof of the theorem will occupy most of the remaining sections of the paper. Before commencing the proof, we state a criterion for the fulfilment of the necessary and sufficient condition involved in the theorem.

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[1] Some of the results of the present paper were included in an abstract entitled *The space of a variable geodesic complex on a sphere*, presented October 25, 1941 to the American Mathematical Society (Bulletin of the Society, vol. 47, 1941, p. 711). Although the precise structure of a space of isomorphic geodesic triangulations of a 2-sphere was there announced, the writer has since noted a certain defect, which remains to be corrected, in that part of his work.

[2] Ernst Steinitz proved a similar theorem for convex polyhedra in euclidean 3-space [E. Steinitz and H. Rademacher, *Vorlesungen über die Theorie der Polyeder*, Berlin (1934), p. 347]. If a convex polyhedron has only triangular faces, its projection from an inner point onto a sphere, S, about the point gives a geodesic triangulation of S. One might deduce the present theorem from that of Steinitz by showing (if it be true) that every geodesic triangulation of S is obtainable as a central projection of a convex polyhedron.

[3] An isomorphism has been defined as a correspondence between vertices, but we use the term also for the homeomorphism between K_0 and K_1 which agrees with the isomorphism and is affine on each simplex in terms of geodesic distances.

Let σ_1, σ_2, σ_3 be any three different 1-cells of K_0 (if such-exist) having a common vertex, P_i, and let σ_1', σ_2', σ_3', P_i' be the respective corresponding cells of K_1. An ordered set of three 1-cells with a common vertex determines an orientation of S.

THEOREM. *If K_0 (and hence K_1) is connected, then a necessary and sufficient condition that it be possible to extend the isomorphism between K_0 and K_1 into an orientation-preserving self-homeomorphism of S is that, for every possible choice of the σ's, the orientation of S determined by $(\sigma_1, \sigma_2, \sigma_3)$ agrees with that determined by $(\sigma_1', \sigma_2', \sigma_3')$.*

This condition was developed in a more general connection by Adkisson and MacLane.[4] The following is a ready extension of the result.

THEOREM. *If K is not connected and K and K' are isomorphic geodesic complexes on S, let $(K_1, \cdots, K_p)$ and $(K_1', \cdots, K_p')$ denote the respective corresponding components of K and K'. The given isomorphism can be extended into an orientation-preserving self-homeomorphism of S if and only if* (1) *the condition of the preceding theorem holds for each pair K_i and K_i' separately and* (2) *as i, j ranges through all ordered pairs of distinct numbers of the set $(1, \cdots, p)$, the components K_j and K_j' are always in regions of $S - K_i$ and $S - K_i'$, respectively, which correspond under an orientation-preserving extension of the given isomorphism as defined for the subcomplexes K_i and K_i'.*

2. Polygonal regions on S

Let T, with vertices $(P_1, \cdots, P_n)$, be a geodesic triangulation of the entire sphere S. Let α_j $(j = 0, 1, 2)$ denote the total number of j-cells in T, and let ν_i $(i = 1, \cdots, \alpha_0 = n)$ denote the number of 1-cells, hence also the number of 2-cells, in the star $S(P_i)$. Then it is easily verified that

$$2\alpha_1 = 3\alpha_2 = \sum_{i=1}^{n} \nu_i . \tag{2.1}$$

The Euler relationship, $\alpha_0 - \alpha_1 + \alpha_2 = 2$, combined with (2.1), gives the well-known identity

$$\sum_{i=1}^{n} (6 - \nu_i) = 12, \tag{2.2}$$

several implications of which will enter into our work. For the present, we state only the following familiar consequence.

(*A*) *There exist vertices of T, each of which is incident with fewer than six 2-cells.*

Consider next a region Δ on S whose boundary, β, is a non-singular 1-circuit made up of geodesic cells. It may be possible to define a geodesic complex covering $\Delta + \beta$ and having just one interior vertex. We will denote the set, perhaps vacuous, of all possible positions for such an interior vertex by Δ^*, and will

[4] V. W. Adkisson and Saunders MacLane, *Extending maps of plane Peano continua*, Duke Mathematical Journal, vol. 6 (1940), pp. 216–218.

refer to Δ^* as the *core* of Δ. It should be recalled that all 1-cells are to be shorter than semi-circles and all 2-cells smaller than hemispheres.

The vertices of β, cyclically numbered, will be denoted by $(R_1, R_2, \cdots)$ and the 1-cells by $s_i = R_iR_{i+1}$ $(i = 1, 2, \cdots)$, where the subscripts are to be taken modulo the number of vertices of β. The cyclic numbering of these vertices determines an orientation of Δ, hence of the entire sphere S. This orientation agrees with the orientation of each of the geodesic 2-cells RR_iR_{i+1} if and only if R is on Δ^*. For each of the values $i = 1, 2, \cdots$, the locus of R such that RR_iR_{i+1} is a geodesic 2-cell with the required orientation consists of the hemisphere, H_i, which has s_i on its boundary and contains a neighborhood on Δ of the mid-point of s_i. Hence Δ^* is the intersection

$$\Delta^* = H_1 \cdot H_2 \cdots \tag{2.3}$$

(*B*) *Let R_i be used to denote not only a vertex of β but also the interior angle of Δ at that vertex. If β has fewer than six vertices and if three consecutive angles R_i are less than π, then $\Delta + \beta$ is a proper subset of some hemisphere.*

Proof. Suppose the notation such that R_1, R_2, and R_3 are all less than π. Then the angle R_2 determines a sector which is the intersection

$$\rho_2 = H_1 \cdot H_2 \tag{2.4}$$

and which has R_1 and R_3 on its boundary. We assume β has five vertices, since the other cases are trivial. Proceeding from R_3 along s_3 or from R_1 along s_5, we enter the sector ρ_2. If s_3 and s_5 do not leave this sector, then Δ is a proper subset of ρ_2. If one of them, say s_3, leaves the sector ρ_2, it cuts therefrom a spherical triangle, namely the intersection $H_1 \cdot H_2 \cdot H_3$, with R_1 on its boundary. Since β is non-singular, s_5 can not leave this triangle. Hence R_5 is on H_2. Also, because s_3 is less than a semicircle, R_4 must be on H_2. Hence Δ is a proper subset of H_2, and β has in common with the boundary of H_2 only the geodesic 1-cell s_2. An appropriate slight rotation carries $\Delta + \beta$ entirely onto H_2 as required.

Lemma 2.1 *If β has fewer than six vertices, then Δ^* is a non-vacuous convex region. If, in addition, Δ is not greater in area than a hemisphere, then some vertex of β is on the boundary of Δ^*.*

Proof. The convexity of Δ^* will follow from (2.3). We can restrict attention to the case where the area of Δ is at most that of a hemisphere, since the contrary case reduces to this one in view of the fact that the cores of Δ and $S - \Delta$ correspond under the antipodal mapping. We also confine ourselves to the case where β has exactly five vertices, for the number of vertices could be increased to five by subdivisions if necessary.

If Δ is at most equal in area to a hemisphere, then Δ possesses at least one pair of non-consecutive angles neither of which exceeds π. For, otherwise, the region $S - \Delta$, whose area is at least that of a hemisphere, would have three consecutive angles less than π in contradiction to result (B).

CASE 1. *No angle of* Δ *exceeds* π. Then $\Delta^* \equiv \Delta$ and our lemma follows at once.

CASE 2. *Just one angle of* Δ *exceeds* π. The vertex of this angle is easily seen to be on the boundary of Δ^*, which is obviously not vacuous.

CASE 3. *Just two angles of* Δ *exceed* π, *and these angles are consecutive.* In this case, one can readily verify that the vertex opposite the edge of β common to these two angles must be on the boundary of Δ^*, and that Δ^* is not vacuous.

CASE 4. *Just two angles of* Δ *exceed* π, *and these angles are not consecutive.* Suppose the numbering is such that the angles R_3 and R_5 exceed π. Then the hemisphere H_1 contains R_3 and R_5 as interior or boundary points. The intersections $\rho_2 = H_1 \cdot H_2$ and $\rho_1 = H_5 \cdot H_1$ are the sectors determined by the interior angles of Δ at R_2 and R_1, respectively. These sectors, since they have s_1 as common edge and both lie on the same side of s_1, intersect in a spherical triangle (or a sector or a hemisphere) with at least one of the points (R_3, R_5) on its boundary. Now the sector $\rho_4 = H_3 \cdot H_4$ has both R_3 and R_5 on its boundary. Therefore Δ^*, which is the intersection $\rho_1\rho_2\rho_4$, must have either R_3 or R_5 on its boundary, and the lemma again follows.

CASE 5. *Just three angles of* Δ *exceed* π. This case, with the above, exhausts the possibilities. The three angles exceeding π cannot be consecutive, since Δ has been shown to have two non-consecutive angles which do not exceed π. We will suppose the notation to be such that R_1, R_2, and R_4 are all greater than π. The sectors $\rho_3 = H_2 \cdot H_3$ and $\rho_5 = H_4 \cdot H_5$, determined by the angles R_3 and R_5 respectively, meet in a region with R_4 on its boundary. Our conclusion will follow if we show that R_4 is on H_1. Now the hemisphere $H_1' = S - \bar{H}_1$ contains R_3 and R_5 as interior points and has R_1 and R_2 on its boundary. If R_4 were on $\bar{H}_1'$, then β would be entirely on $\bar{H}_1'$. But Δ would then contain all of H_1 and part of H_1', in contradiction to the fact that it does not exceed a hemisphere in area. This completes the proof of Lemma 2.1.

LEMMA 2.2 *As* β *varies continuously on* S, *remaining isomorphic to its original position, the region* Δ *and its core* Δ^* *also vary continuously, provided* β *has fewer than six vertices. It is then possible to define a homeomorphism between* Δ^* *and a fixed* 2*-cell,* s_0^2, *in such a way that the image of any point on* s_0^2 *varies continuously with* β.

PROOF. Let s_0^2 be the interior of the unit circle about the pole on a plane supplied with a polar coordinate system (ρ, ϑ). The center of s_0^2 shall correspond to the end-point, C, on Δ^* of the radius of S through the centroid of Δ^*, regarded as a surface in the euclidean 3-space of S. Each radius of s_0^2 shall correspond to a geodesic 1-cell from C to the boundary of Δ^*, and corresponding points on these 1-cells shall divide them in the same ratio. Angles at the center of s_0^2 shall correspond to equal angles at C, with $\vartheta = 0$ corresponding to CR_1 and with ϑ increasing as the edge s_1 of β is traced from R_1 to R_2. This completely specifies a homeomorphism satisfying the lemma.

LEMMA 2.3 *Suppose* β *has fewer than six vertices. Given an arbitrary, but fixed, subset* $[R]$ *of the vertices of* β, *let* D *be the set of all points on* Δ *each of which*

can be joined to every point of $[R]$ *by a geodesic* 1*-cell entirely on* Δ. *Then* D *is a non-vacuous region, star-shaped from every point of the core,* Δ^*, *of* Δ.

PROOF. If $[R]$ is vacuous, then $D \equiv \Delta$ and the result follows from the definition of Δ^*. In any case, we clearly have $D \supset \Delta^*$, so that D cannot be vacuous, in view of Lemma 2.1. Given any vertex, R_i, of β, let $V(R_i)$ denote the set of all points on Δ each of which can be joined to R_i by a geodesic 1-cell entirely on Δ. Then D is the intersection of all the regions $V(R_j)$ as R_j ranges over the set $[R]$. The above lemma is thus equivalent to the following statement: *If* P *is on* $\overline{V(R_j)}$ *and* P^* *is on* Δ^*, *then the geodesic* 1*-cell* PP^* *is on* $V(R_j)$. We may as well take P to be a boundary point of $V(R_j)$. The interior of the spherical triangle P^*PR_j is on Δ; for R_jP^* and P^*P are both on Δ and R_jP is on $\bar{\Delta}$. Hence every geodesic 1-cell from R_j to a point of P^*P is on Δ, and our conclusion is immediate.

(C) *Lemma* 2.2 *above holds with the region* D *of Lemma* 2.3 *replacing* Δ^*.

3. Geodesic representations of complexes

LEMMA 3.1 *Let* C *be any non-singular homogeneously* 2*-dimensional simplicial complex on* S, *with at least two* 2*-cells. Suppose* C *is strongly connected; that is, that any two of its* 2*-cells can be joined by a sequence of* 2*-cells, in which each consecutive pair have a bounding* 1*-cell in common. Suppose further that every* 1*-cell with both its vertices on the boundary of* C *lies entirely on that boundary. Then* C *is isomorphic with some geodesic complex,* K, *on* S, *such that* $S - K$ *consists of a finite number of distinct convex regions, with no three boundary vertices on a great circle.*

PROOF. Our hypotheses imply that C has at least one inner vertex, P. Let the star $S(P)$ on C be represented as a star of geodesic cells, $S_0(P)$, in such a way that $S - \overline{S_0(P)}$ is a convex region with no three boundary vertices all on the same great circle. This determines the position, as a geodesic cell, of an image for every cell of C whose vertices are all on the boundary of $S(P)$. Let these geodesic images be added to $S_0(P)$ to obtain a geodesic mapping, K_1, of a subcomplex, C_1, of C. This serves as initial step for a recurrent process based on the following hypothesis: For some value $j > 0$, a geodesic complex, K_j, has been defined isomorphic to a certain subcomplex, C_j, of C containing at least j 2-cells. Furthermore, $S - K_j$ is the sum of a finite number of convex regions, none of which has three vertices all on the same great circle. If C_j is a proper subcomplex of C, there exists a boundary vertex, P', of C_j which is not a boundary vertex of C. The typical step of the recurrency commences with the addition to K_j of a geodesic mapping of that part of the star $S(P')$ which is not on C_j. This can easily be done so as to preserve the above convexity condition. One then inserts, in the positions determined by the mapping, a geodesic image for every cell of $C - [C_j + S(P')]$ with all its vertices on $\overline{C_j + S(P')}$. This defines K_{j+1}, with the required properties.

COROLLARY. *Any non-singular simplicial complex,* C, *on* S *is isomorphic to some non-singular geodesic complex on* S.

For, C can be regarded as a subcomplex of some triangulation T of the entire sphere, and Lemma 3.1 can then be applied to T.

4. Proof of the deformation theorem

LEMMA 4.1 *Let K_0 and K_1 be two geodesic complexes on S which correspond under an isomorphism extensible into an orientation-preserving self-homeomorphism of S. Then there exist two geodesic complexes, T_0 and T_1 which cover S, which have K_0 and K_1 as respective subcomplexes, and which correspond under an orientation-preserving isomorphism agreeing on K_0 and K_1 with the given isomorphism.*

This lemma follows from known properties of 2-dimensional regions.

(*A*) *As a direct consequence of Lemma* 4.1, *the proof of the deformation theorem in the general case reduces to the proof for a pair of geodesic triangulations, T_0 and T_1, of the entire sphere S.*

The proof for the complexes T_0 and T_1 will be by a recurrency. The initial step consists in the easy verification of the result in the case where T_0 has just four vertices. The recurrency is based on the following hypothesis.

HYPOTHESIS. *For some value $n > 4$, it is known that the deformation theorem holds for geodesic triangulations of S having at most $(n - 1)$ vertices.*

A geodesic complex, K, with vertices $(P_1, \cdots, P_n)$, will be called *reducible* if the two end-points (P_i, P_j) of some 1-cell can be moved continuously into coincidence, all other vertices remaining fixed, in such a way (1) that no cell of K becomes degenerate until P_i and P_j reach coincidence and (2) that when P_i and P_j reach coincidence, the only cells which become degenerate are those with both P_i and P_j on their boundaries. When P_i and P_j are thus made to coincide, there results a geodesic complex K' with $n - 1$ vertices.

LEMMA 4.2 *Every geodesic complex T which covers S and has at least five vertices is reducible.*

PROOF. As in §2, we denote by $(P_1, \cdots, P_n)$ the vertices of T and by ν_i $(i = 1, \cdots, n)$ the number of 2-cells in the star $S(P_i)$.

CASE 1. *There exist two vertices, P_i and P_j, joined by a 1-cell of T, where $\nu_i + \nu_j < 10$.*

(*B*) *It is always possible, under the hypotheses of this case, further to restrict i and j so that the cells on $S(P_i) + S(P_j)$ will cover a polygonal region, Δ, whose boundary, β, is a non singular 1-circuit made up of $\nu_i + \nu_j - 4 < 6$ geodesic 1-cells.*

To establish (B), we first note that $\nu_i = \nu_j = 3$ would imply that T has but four vertices, contrary to the hypotheses of the present lemma. If $\nu_i = 3$ or $\nu_j = 3$, then (B) is automatically fulfilled, but this is not so if $\nu_i > 3$ and $\nu_j > 3$. If the restriction fails with $\nu_i = \nu_j = 4$, then $S(P_i) + S(P_j)$ covers the whole sphere, and T is structurally a tetrahedron triangulated by the introduction of a vertex in the edge opposite P_iP_j. If P' is one of the two vertices incident with both P_i and P_j, then we have merely to let P' play the role of P_i, retaining P_j, in order to satisfy (B). The only remaining case is, essentially, $\nu_i = 4$, $\nu_j = 5$; this being equivalent to $\nu_i = 5$, $\nu_j = 4$. Let the vertices of $S(P_i)$, cyclically numbered, be denoted by (P', P_j, P'', A); and the vertices of $S(P_j)$,

also cyclically numbered, by (P', P_i, P'', B_1, B_2). If A coincides with neither B_1 nor B_2, then (B) is fulfilled. If $A \equiv B_2$, then P' can be used in place of P_j, retaining P_i, to fulfill (B); and, if $A \equiv B_1$, then P'' can similarly be used in place of P_j.

Assume restriction (B). We next modify T as follows, to obtain a complex, T', with just $n - 1$ vertices. The cells on $S(P_i)$ and $S(P_j)$, which cover Δ, are first deleted. The region Δ is then covered by the star, $S(P)$, of a single vertex, where P is any point of the core Δ^* of Δ, and $S(P)$ is the triangulation of Δ obtained by joining P to the vertices of β by geodesic 1-cells.

We complete the present case of Lemma 4.2 by showing that P_i and P_j can be moved into the common position P in accordance with the definition of reducibility. Consider the region D of Lemma 2.3, applied to the Δ and β of our present argument, with $[R]$ denoting those vertices of β which are joined to P_i by 1-cells of T. As P_i varies on D, all other vertices of T being fixed, the geodesic cells of T determined by P_i and points of $[R]$ remain non-degenerate. The region, Δ_j, covered by $S(P_j)$ in the original position of the vertices varies continuously with P_i. Hence, by Lemma 2.2, as P_i moves continuously on D, P_j can be made to move continuously so as to remain always on the variable core, Δ_j^*, of Δ_j; all vertices of T other than P_i and P_j being fixed. The first phase of our reduction of T is a motion of P_i along a geodesic arc into P [cf. Lemma 2.3], while P_j varies continuously and stays on Δ_j^*. The second phase is a motion of P_j into the position $P \equiv P_i$ along a geodesic arc, other vertices remaining fixed. It is easily seen that the entire motion satisfies the requirements in the definition of a reducible complex, since (1) P_i remains on D and (2) P_j remains on the convex region Δ_j^* until it reaches coincidence with the boundary vertex $P_i \equiv P$ of Δ_j^*.

CASE 2. *The condition of Case* 1 *is not fulfilled.* Since $6 - \nu_i$ cannot exceed 3, the identity (2.2) implies that at least four of the numbers ν_i are less than 6. If one of these numbers, ν_j, is less than 5, and if $\nu_i < 6$ $(i \neq j)$, then the stars $S(P_j)$ and $S(P_i)$ must be distinct, since the condition of Case 1 does not hold. If none of the numbers ν_i is less than five, then, by (2.2), at least twelve of them must be equal to five. Hence it is readily seen that, under the present case, there must be at least two vertices whose stars are distinct and contain fewer than six 2-cells each. One of these, $S(P_j)$, must therefore cover a region, Δ, smaller than a hemisphere. The core Δ^* is the locus of all positions to which P_j can be moved, the other vertices of T remaining fixed, without having any cell become degenerate. By Lemma 2.1, some vertex, P_i, is on the boundary of Δ^*. Hence P_i and P_j can be moved into coincidence as required by the present lemma.

(C) *It remains to prove, on the basis of the above hypothesis, that the deformation theorem holds as applied to a pair, T_0 and T_1, of geodesic triangulations of S, where T_0 (hence also T_1) has just n vertices.*

We work under the same division into cases as in the proof, just completed, of Lemma 4.2.

CASE 1. Let $(P_1, \cdots, P_n)$ be the vertices of T_0, and let $(Q_1, \cdots, Q_n)$

be the vertices of T_1, similarly numbered. Then T_0 is reducible to a complex T_0' by a motion of P_i and P_j [see preceding proof for notation] into a common position, P_{ij}, on Δ^*, and T_1 is reducible to a complex T_1' by a motion of Q_i and Q_j into a position Q_{ij}. By the hypothesis of the recurrency, T_0' can be deformed isotopically into T_1' as described in the deformation theorem. During this deformation, the boundary, β, of $S(P_{ij})$ varies continuously. Let Δ be the region covered by $S(P_{ij})$, and let $[R]$ be the set of vertices of $S(P_i)$ on β in the original complex T_0. The corresponding domain D of Lemma 2.3 contains P_i in its original position. Let §2(C) be applied, and let P_i vary during the deformation so as to keep a fixed image under a homeomorphism of the sort involved in §2(C). When the final position has been reached, let P_i be moved into Q_i along a continuous curve on D [cf Lemma 2.3]. We thus obtain an isotopic deformation, of the sort described in the deformation theorem, carrying $T_0 - S(P_j)$ into $T_1 - S(Q_j)$. To extend it to a deformation of T_0 into T_1, we apply Lemma 2.2 to the varying region Δ_j originally covered by $S(P_j)$. Throughout the deformation, we let P_j retain a fixed image under the homeomorphism of Lemma 2.2. When the deformation is completed, Δ_j becomes the region covered by $S(Q_j)$, and both P_j and Q_j are on the core Δ_j^*. Hence it remains only to move P_j along a geodesic 1-cell into coincidence with Q_j, as permitted by the convexity of Δ_j^* [Lemma 2.1]. The deformation theorem is thus seen to hold for complexes with n vertices subject to the hypothesis of Case 1.

CASE 2. LEMMA 4.3 *If the condition of Case* 1 *does not hold, there exist three vertices* (P_h, P_j, P_k) *no two of whose stars have a* 2*-cell in common and each of whose stars contains fewer than six* 2*-cells.*

To prove this, suppose first that there are two numbers, ν_h and ν_j, which are both less than five. Then if $\nu_k < 6$, the three points (P_h, P_j, P_k) satisfy the requirements of Lemma 4.3, since the conditions defining Case 1 are not fulfilled. If there is just one number, ν_h, less than five, then, by equation (2.2), there must be at least nine others of the ν_i equal to five. Among these nine or more, we need only select a pair (ν_j, ν_k) such that $S(P_j)$ and $S(P_k)$ are distinct, in order to satisfy the requirements of our lemma. If none of the ν_i are less than five, then it is easy, from among the twelve or more which must equal five, to select three (ν_h, ν_j, ν_k) so that the stars of (P_h, P_j, P_k) shall be distinct.

The isomorphism between T_0 and T_1 carries Lemma 4.3 over to T_1. At least two of the stars $[S(P_h), S(P_j), S(P_k)]$ and at least two of the stars $[S(Q_h), S(Q_j), S(Q_k)]$ must obviously cover regions less in area than a hemisphere. Let the notation be selected so that both $S(P_h)$ and $S(Q_h)$ have this property. Let Δ_0 denote the region covered by $S(P_h)$ and let Δ_0^* be its core. By slight displacements, if necessary, arrange that no three boundary vertices of $S(P_h)$ be on a great circle. Since some boundary vertex, P_r, of Δ_0 is on the boundary of Δ_0^*, by Lemma 2.1, the complex T_0 is reducible to a complex T_0' by a motion of P_h into P_r. Now the complex T_0 and the complex T_0' have in common the subcomplex $K_0 = T_0 - S(P_h)$. Let β_0 be the boundary of K_0.

SUBCASE *a*. It is convenient to divide the remainder of the proof into two parts, imposing first a simplifying hypothesis on the structure of K_0, so as to bring out the nature of the reasoning. We accordingly suppose that no two vertices of β_0 are joined by an inner 1-cell of K_0. Lemma 3.1 then assures the existence of a geodesic complex, K, which corresponds to K_0 under an orientation-preserving isomorphism and has the property that the region $\Delta = S - K$ is convex. By introducing geodesic 1-cells joining the boundary vertex of Δ corresponding to P_r to the other boundary vertices, we can triangulate Δ so as to extend K into a complex, T', isomorphic to T_0'. It is then possible, by the hypothesis of the recurrency, to carry T_0' into T' in accordance with the requirements of the deformation theorem. This induces a deformation of K_0 into K, during which the region Δ_0 varies continuously, so that Lemma 2.2 applies. Hence P_h can remain on the core Δ_0^* and vary continuously throughout the deformation. By the definition of a core (§2), this means that the original complex T_0 can be deformed isotopically into a complex T which has K for a subcomplex. In similar fashion, T_1 can also be deformed into a complex T^* which has K for a subcomplex. But T and T^* are then identical save for their respective vertices corresponding to P_h. These, however, can be moved into coincidence along the geodesic arc joining them, since the core of the region Δ is convex [Lemma 2.1]. This completes a deformation of T_0 and T_1 into a common position and thus completes a proof of the deformation theorem under the present restriction on the structure of K_0.

SUBCASE *b*. Suppose, finally, that there exist vertices of β_0 joined by inner 1-cells of K_0. The complex T_0 results from K_0 when the region $\Delta_0 = S - K_0$, bounded by β_0, is geodesically triangulated by 1-cells from P_r to the other vertices of β_0. Hence the following statement.

(*D*) *The vertex P_r is not joined by any inner 1-cell of K_0 to any other vertex of β_0.*

Among the 1-circuits of K_0 having all their vertices on β_0, let β_0' be the one with the fewest cells. Then β_0' is obtainable by replacing certain cells of β_0 by inner 1-cells of K_0 which join non-consecutive vertices of β_0. Let Δ_0' be the region on S bounded by β_0' and containing Δ_0; and let K_0' be the subcomplex of K_0 consisting of all the cells thereof on the region $S - \Delta_0'$. Then if K_0' possesses more than one 2-cell, the hypotheses of Lemma 3.1, read for $C = K_0'$ are satisfied. Hence there exists a geodesic complex, K', corresponding to K_0' under an orientation-preserving isomorphism and having the property that the region $\Delta' = S - K'$ is convex. If K' possesses just one 2-cell, then $\Delta' = S - K'$ is the region complementary to a closed geodesic 2-cell.

(*E*) *In either case, by appropriately filling in part of the region Δ', we can extend K' into an isomorphic image, K, of $K_0 = T_0 - S(P_h)$. This filling in can be done so that, if $\Delta = S - K$, then Δ can be geodesically triangulated by 1-cells from the vertex corresponding to P_r.*

To establish (E), let it first be noted that the image of P_r cannot, in view of result (D), be incident with any bounding 1-cell of Δ' which is not also a bounding 1-cell of Δ. The result (E) can then be readily verified by an adaptation of the

proof of Lemma 3.1. In carrying through the details, one should first insert the bounding 1-cells of Δ not on K' in such a way as to assure the required triangulability of Δ.

In subcase a, the only purpose in making Δ convex was to assure the possibility of geodesically triangulating it by 1-cells from the vertex corresponding to P_r. In the present subcase, this possibility is assured by (E). The proof can now be completed, with slight modifications, as in subcase a.

This terminates the typical step of our recurrent process and thus completes an establishment of the deformation theorem on the sphere.

5. Deformations of plane rectilinear complexes

DEFORMATION THEOREM. *Let K_0 and K_1 be a pair of finite isomorphic rectilinear complexes on a bounded region of a euclidean plane, E. Then if and only if the isomorphism between K_0 and K_1 can be extended into an orientation-preserving self-homeomorphism of E, there exists an isotopic deformation of K_0 into K_1 induced by a motion of the vertices into their respective images, all cells remaining rectilinear and non-degenerate throughout the motion.*

PROOF. For the most part, the proof of the present deformation theorem consists in reading §§2, 3, and 4, replacing (1) S by E (2) "geodesic" by "rectilinear" (3) "hemisphere" by "half-plane." Certain simplifications result from the fact that in the present case no difficulties arise analogous to those which result when certain regions on the sphere are not proper subsets of a hemisphere. We outline the principal changes other than the replacements (1), (2), and (3) just listed.

We temporarily postpone consideration of §2. In the rectilinear form of Lemma 3.1, the unbounded component of $E - K$ cannot be convex. We can however require that the outer boundary of K be a convex polygon and that all bounded components of $S - K$ be convex, no three boundary vertices being collinear. The recurrent proof can then commence with any isomorphic mapping of the outer boundary of C onto a convex polygon and the introduction of simplexes corresponding to those inner cells of C whose vertices are all on the outer boundary of C.

The statement of Lemma 4.1 can be changed by letting the complexes T_0 and T_1 be rectilinear simplicial subdivisions of a pair of triangles, $P_1P_2P_3$ and $Q_1Q_2Q_3$ respectively, and it can be required that T_0 and T_1 not subdivide the edges of the triangles. Then we can move $P_1P_2P_3$ into coincidence with $Q_1Q_2Q_3$, keeping fixed the barycentric coordinates of each vertex of T_0 relative to the moving triangle. This leads to the following replacement of §4(A).

(*A*) *The proof of our deformation theorem on the plane in the general case reduces to the proof for a pair of rectilinear triangulations, T_0 and T_1, of a fixed triangular region, $P_1P_2P_3$, where T_0 and T_1 are isomorphic with P_i self-corresponding (i = 1, 2, 3), and where all vertices save (P_1, P_2, P_3) are inner vertices. The vertices (P_1, P_2, P_3) can be kept fixed throughout the required deformation.*

In place of Lemma 4.2, we have the following.

LEMMA 5.1. *A rectilinear triangulation, T, of $P_1P_2P_3$ with no boundary vertices save (P_1, P_2, P_3) is always reducible, except in the trivial case where T has no inner vertices.*

In proving this result, we can use identity (2, 2), provided we count the region $E - T$ as a 2-simplex incident with P_1, P_2, and P_3. We immediately deduce the existence of a star $S(P_j)$ $(j > 3)$ for which $\nu_j < 6$. The last three sentences in the proof of Lemma 4.2 will then complete the present argument; since Lemma 2.1, with a simplified statement and proof, holds in the planar case. In Lemmas 2.1 and 4.2, the spherical case is relatively complicated by the possibility of some of the regions Δ not being proper subsets of hemispheres, even when they are smaller in area than hemispheres. The planar form of Lemma 2.1 could be obtained from the spherical by a central projection of a hemisphere onto a tangent plane. Case 5 in the proof of Lemma 2.1 cannot occur on the plane.

The planar forms of Lemmas 2.2, 2.3, and of §2(C) all hold, and they can be proved by arguments precisely analogous to those used in the spherical case.

It remains to consider the recurrent process establishing the deformation theorem and commencing just after §4(A). In the present proof, the initial step deals with the trivial case where T_0 has no inner vertices [cf §5(A)]. The recurrent hypothesis should now be read with $n > 3$ in place of $n > 4$. The general step, commencing with §4(C), is directly adaptable to the rectilinear case. A considerable simplification results from the fact that nothing analogous to Lemma 4.3 is needed, because Lemma 4.3 merely assured the possibility of choosing h so that $S(P_h)$ and $S(Q_h)$ would each be less in area than a hemisphere.

QUEENS COLLEGE, NEW YORK.

ANNALS OF MATHEMATICS
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INTRODUCTION OF A RIEMANNIAN GEOMETRY ON A TRIANGULABLE 4-MANIFOLD[1]

BY STEWART S. CAIRNS

(Received February 9, 1943)

In this note, the writer completes the establishment of the following result.

THEOREM. *If M is a triangulable m-manifold, $m \leqq 4$, then there exists an analytic Riemannian manifold homeomorphic to M.*

Since every analytic, indeed every differentiable, manifold is known to be triangulable,[2] this theorem permits us to say that the classes of triangulable m-manifolds and of analytic Riemannian manifolds are topologically equivalent ($m \leqq 4$) in the sense that an arbitrary member of either class can be mapped topologically on some member of the other class.

A topological equivalence theorem for two classes of manifolds, or other objects, one of which has a more restricted definition than the other, is of obvious potential usefulness, for it may carry over to the less restricted class certain results previously proved for the more restricted. For an example, note the application below of a topological equivalence theorem due to Whitehead.

In an earlier paper,[3] the writer obtained sufficient conditions for the possibility of introducing an analytic Riemannian geometry on a topological manifold of any dimensionality [H, Theorems I and V]. For the case of a triangulable m-manifold, $m \leqq 3$, the fulfillment of these conditions is there verified [H, Theorem III and §11]. For the case of a 4-dimensional Brouwer manifold, the problem of verifying these conditions is reduced to a certain deformation problem [H, p. 808] pertaining to geodesic triangulations of a 2-sphere. The writer has since generalized and solved this problem.[4] Hence the present theorem is justified with the restriction, in the case $m = 4$, that M have a Brouwer triangulation. This restriction, however, can be removed, as a consequence of a recent theorem due to Whitehead.[5] Although non-Brouwer triangulated manifolds had been shown to exist,[6] Whitehead proved that every triangulated manifold can be

[1] Presented in part to the American Mathematical Society, October 25, 1941, under the title *Topological mapping of a Brouwer 4-manifold on an analytic Riemannian 4-manifold* (Bulletin of the Society, vol. 47, 1941, p. 711).

[2] S. S. Cairns, *Triangulation of the manifold of class one*, Bulletin of the American Mathematical Society, vol. 41 (1935), pp. 549–552.

[3] S. S. Cairns, *Homeomorphisms between topological manifolds and analytic Riemannian manifolds*, these Annals, vol. 41 (1940), pp. 796–808. This paper is hereafter referred to as H.

[4] *Isotopic deformations of geodesic complexes on the 2-sphere and on the plane*, these Annals, immediately preceding the present article.

[5] J. H. C. Whitehead, *Note on manifolds*, Quarterly Journal of Mathematics, Oxford series, 12 (1941), pp. 26–29.

[6] S. S. Cairns, *Triangulated manifolds which are not Brouwer manifolds*, these Annals, vol. 41 (1940), pp. 792–795. This reference also contains definitions of all the types of manifolds involved in the present discussion. In the terminology there used, we note that

subdivided into a Brouwer manifold. Hence statement (A) at the end of reference H holds for all triangulable manifolds, and the argument is complete, in view of the solution of the deformation problem.[4]

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Whitehead's theorem establishes the topological equivalence of triangulable, star, and Brouwer manifolds, reducing the writer's hierarchy of five classes of manifolds to topological, triangulable ($\simeq$ star $\simeq$ Brouwer) and differentiable ($\simeq$ analytic $\simeq$ analytic Riemannian), where $\simeq$ indicates topological equivalence. All the classes are known to be equivalent for dimensions less than three, and the triangulable are now known to be equivalent to the differentiable for dimensions less than five.

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THE SELF-INTERSECTIONS OF A SMOOTH n-MANIFOLD IN $2n$-SPACE*

BY HASSLER WHITNEY

(Received August 19, 1943)

1. Introduction

Let f be a regular mapping (see the definitions below) of the simple closed curve M^1 into the plane E^2. The resulting curve $C = f(M)$ may cut itself a number of times; if this number is finite, and the "positive" and "negative" self-intersections are distinguished, the algebraic number I_f of them is invariant under "regular deformations," which keep the mapping always regular.[1] We may determine I_f by considering the space $\mathfrak{T}$ of ordered pairs of points of M^1, mapping it into E^2 in a manner determined by f (see below), and counting the coverings of the origin. The object of this paper is to study the situation in n dimensions, for regular mappings of a manifold M^n into E^{2n}. The main theorem (which is trivial for $n = 1$ and well known for $n = 2$) is that M^n may be imbedded in E^{2n}.

An outline of the paper follows. Take $\mathrm{E}^n \subset E^{2n}$. It is shown how E^n may be distorted near the origin, giving a mapping $f(E^n) \subset E^{2n}$ such that there is just one self-intersection; specific equations (2.2) for f are given. By introducing such self-intersections into the mapping f of a manifold M^n into E^{2n}, we may alter I_f as we please. As in the case $n = 1$, we express I_f in terms of a mapping F of $\mathfrak{T}$ into V^{2n} = all vectors in E^{2n}, provided that n is even and M is orientable. In any other case, I_f is only determined mod 2; we must identify (p, q) with (q, p) in $\mathfrak{T}$ $(p, q \in M)$, forming a space $\mathfrak{T}^*$, and let F^* map it into V^*, formed from V by identifying each vector with its negative. The determination of the "divided degree" needed is accomplished through a study of "locally derivable" mappings F^* of spaces into V^*; these are obtainable, locally, as the projection into V^* of mappings into V. We carry this theory somewhat further (in §§5, 6) than is needed in the present paper; the notions are capable of great generalization.

After the proof of the invariance of I_f under regular deformations comes the key theorem (Theorem 4) in the proof of the imbedding theorem; it says essentially that the number of self-intersections may be reduced by two, if the number present is greater than $|I_f|$ (or is $\geqq 2$, if I_f is taken mod 2). It is a curious fact that this theorem is proved only in the case $n \geqq 3$. The proof of the imbedding theorem is now immediate if M is closed, and is easily carried out if M is open. In an appendix, we prove that the complex projective plane (a manifold of di-

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mension 4) may be imbedded in E^7; since[2] its characteristic is odd, it cannot be imbedded in E^6.

We give a brief description of $\mathfrak{T}$ and F. The set of all pairs (p, u), $p \in M$, u a unit vector tangent to M at p, is the "tangent space" $\mathfrak{S}$ of M. The space $\mathfrak{T}$ consists of $\mathfrak{S}$, and all pairs (p, q), $p \neq q$ $(p, q \in M)$; if $q \to p$ in the direction of a unit vector u at p, we let $(p, q) \to (p, u)$. Thus $\mathfrak{T}$ is bounded by $\mathfrak{S}$. F maps (p, q) into a multiple of $f(q) - f(p)$, so chosen that $\mathfrak{S}$ is mapped into vectors $\neq 0$.

NOTATIONS. A mapping of a manifold M^n is *smooth* if it is of class C^1 (has continuous first partial derivatives). It is *regular* if the Jacobian matrix is everywhere of rank n, or, n independent vectors at any point are carried into n independent vectors, or, each non-zero vector is carried into a non-zero vector. It is an *imbedding* if it is proper (see §24 of the following paper; we may omit this if M is closed) and regular, and is one-one; it is an *immersion* if it is proper and regular. Let I denote the unit interval $0 \leqq x \leqq 1$. We use ∂ for boundary in both the point set and combinatorial sense. If f maps chains in the complex K into chains in K', its dual f' is defined by

$$f'(\sum a_i' \sigma_i') = \sum_j a_i' \alpha_{ji} \sigma_j, \quad \text{where} \quad f(\sigma_i) = \sum_j \alpha_{ij} \sigma_j'.$$

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I. COMBINATORIAL STUDY OF SELF-INTERSECTIONS

2. A particular self-intersection

We shall define a regular mapping f of class C^∞ of the interior Q^n of the unit sphere in E^n into E^{2n} such that there is just one self-intersection, and so that f equals the identity, with all derivatives, on the boundary of Q^n.

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For $n = 1$, a curve with one loop, like the written letter e, gives the required self-intersection. Such a mapping of x-space into (y, z)-space may be expressed in the form

$$y = x - \frac{2x}{1 + x^2}, \qquad z = \frac{1}{1 + x^2}. \tag{2.1}$$

[2] H. Seifert, *Algebraische Approximationen von Mannigfaltigkeiten*, Math. Zeitschrift, vol. 41 (1936), pp. 1–17, Satz 2; compare E. Stiefel, *Richtungsfelder und Fernparallelismus in n-dimensionalen Mannigfaltigkeiten*, Comp. Math. Helvetici, vol. 8 (1936), Anhang II; also [3].

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For $x = \pm 10$ say, these are approximately $y = x$, $z = 0$; hence we may flatten out $f(M)$ near $x = \pm 10$, to obtain exactly these equations there. For general n, the example will again be of a mapping near the identity for large values of the x's, and we shall not trouble to mention the final alterations necessary. (The detailed proof may be given with the help of Lemma 11 of the following paper.)

Consider E^4 as given by moving an E^3 parallel to itself in the y_2-direction. Take $E^2 \subset E^3 \subset E^4$, and take the above $f(M^1) \subset E^2$. As E^3 moves, say with increasing y_2, let the two parts of $f(M^1)$ near the self-intersection be pulled apart, into opposite sides of E^2 in E^3. As y_2 continues to increase, the curve may be flattened out into the line parallel to the y_1-axis. If we let y_2 decrease from 0, we pull the two parts of $f(M^1)$ away in the opposite directions, etc. This clearly defines a self-intersection of the required type. The case $n = 2$ in (2.2) is easily seen to define this mapping.

For general n, the equations are[3]

$$u = (1 + x_1^2) \cdots (1 + x_n^2);$$

$$y_1 = x_1 - \frac{2x_1}{u}, \qquad y_i = x_i \qquad (i = 2, \cdots, n), \tag{2.2}$$

$$y_{n+1} = \frac{1}{u}, \qquad y_{n+i} = \frac{x_1 x_i}{u} \qquad (i = 2, \cdots, n).$$

The matrix of partial derivatives, for example for $n = 3$, is, transposed,

$$\begin{Vmatrix} 1 - \dfrac{2(1 - x_1^2)}{u(1 + x_1^2)} & 0 & 0 & \dfrac{-2x_1}{u(1 + x_1^2)} & \dfrac{x_2(1 - x_1^2)}{u(1 + x_1^2)} & \dfrac{x_3(1 - x_1^2)}{u(1 + x_1^2)} \\ \dfrac{4x_1 x_2}{u(1 + x_2^2)} & 1 & 0 & \dfrac{-2x_2}{u(1 + x_2^2)} & \dfrac{x_1(1 - x_2^2)}{u(1 + x_2^2)} & \dfrac{-2x_1 x_2 x_3}{u(1 + x_2^2)} \\ \dfrac{4x_1 x_3}{u(1 + x_3^2)} & 0 & 1 & \dfrac{-2x_3}{u(1 + x_3^2)} & \dfrac{-2x_1 x_2 x_3}{u(1 + x_3^2)} & \dfrac{x_1(1 - x_3^2)}{u(1 + x_3^2)} \end{Vmatrix}.$$

To show that f is regular, we must find, for each $p = (x_1, \cdots, x_n)$, a determinant of order n which is $\neq 0$. It is clearly sufficient to find an element $\neq 0$ in the first row. Suppose the element $\partial y_{n+1}/\partial x_1$ $(= \partial y_4/\partial x_1)$ is 0. Then $x_1 = 0$, and hence $\partial y_1/\partial x_1 = 1 - 2/u$. If this is 0, then $u = 2$; hence not all the x_i are 0, and (since $x_1 = 0$) some $\partial y_{n+i}/\partial x_1$ is $\neq 0$.

Next we find the self-intersections. Suppose

$$f(x_1, \cdots, x_n) = f(x_1', \cdots, x_n'), \qquad (x_1, \cdots, x_n) \neq (x_1', \cdots, x_n').$$

Since $y_i' = y_i$ $(i = 2, \cdots, n)$, $x_i' = x_i$ for $i > 1$. Since $y_{n+1}' = y_{n+1}$, $u' = u$; it follows that $x_1'^2 = x_1^2$; hence $x_1' = -x_1 \neq 0$. Since $y_{n+i}' = y_{n+i}$ $(i > 1)$, $x_i =$

[3] The equations are seen to be closely related to those defining the singularity (3.3) of the following paper. It would be possible to define the required self-intersections and singularities in turn, mapping E^1 into E^2, E^2 into E^3, E^3 into E^4, E^4 into E^5, etc., using each time the preceding mapping.

0 $(i > 1)$. Since $y_1' = y_1$,

$$x_1 - \frac{2x_1}{u} = -x_1 + \frac{2x_1}{u}, \qquad u = 1 + x_1^2 = 2,$$

and $x_1 = \pm 1$. Thus the only self-intersection is

$$f(1, 0, \cdots, 0) = f(-1, 0, \cdots, 0).$$

To show that this is a regular intersection, we note that the matrix $\| \partial y_i / \partial x_j \|$ transposed, at $(\pm 1, 0, \cdots, 0)$, is (for instance for $n = 3$)

$$\begin{Vmatrix} 1 & 0 & 0 & \mp\frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & \pm\frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & \pm\frac{1}{2} \end{Vmatrix}.$$

The elements of the i^{th} row are the components of the vector $\partial f / \partial x_i$ at $(\pm 1, 0, \cdots 0)$. The two matrices with the two sets of signs give a $2n$-rowed square matrix whose determinant we must prove $\neq 0$. If we subtract the i^{th} row from the $(n + i)^{\text{th}}$ row and expand in terms of the first n columns, we obtain a determinant with diagonal terms equal to 1 or -1, and zeros below the diagonal; hence it is $\neq 0$.

Finally, in the boundary of the region $\sum x_i^2 \leqq 100n$ for example, the equations are practically

$$y_i = x_i \ (i = 1, \cdots, n), \qquad y_i = 0 \ (i > n);$$

hence we may flatten out the mapping on the boundary.

3. The spaces $\mathfrak{T}$ and $\mathfrak{T}^*$

Consider M^n as lying in E^{2n+1} (see [1], Theorem 1). The points of $\mathfrak{S}$ are the pairs (p, v), $p \in M$, v a unit vector tangent to M at p. The points of $\mathfrak{T} - \mathfrak{S}$ are the pairs (p, q), $q \neq p$. We define neighborhoods in $\mathfrak{T} - \mathfrak{S}$ in the obvious fashion. To define neighborhoods in $\mathfrak{T}$ at points of $\mathfrak{S}$, we first define a subsidiary space $\mathfrak{T}'$ as follows. Choose a positive continuous function $\eta(p)$ (which may be made constant if M is compact) such that for each $p \in M$ and sphere $S(p)$ of radius $\eta(p)$ in the tangent plane $T(p)$ to M at p, $S(p)$ plus interior projects (along the planes perpendicular to $T(p)$) onto M in a one-one way. Let π_p be this projection. The points of $\mathfrak{T}'$ are the triples (p, v, λ), $p \in M$, v is a unit vector tangent to M at p, and $0 \leqq \lambda < \eta(p)$. The definition of neighborhoods in $\mathfrak{T}'$ is clear. We map $\mathfrak{T}'$ onto a portion of $\mathfrak{T}$ by setting

$$\Phi(p, v, 0) = (p, v) \in \mathfrak{S},$$

$$\Phi(p, v, \lambda) = (p, \pi_p(p + \lambda v)) \in \mathfrak{T} - \mathfrak{S} \qquad (\lambda > 0).$$

Neighborhoods in $\mathfrak{T}'$ map into neighborhoods in $\mathfrak{T}$.

If M is a manifold of class C^r, $\mathfrak{T}$ is a manifold with boundary, of class C^{r-1}. Clearly the choice of the imbedding of M^n in E^{2n+1} does not affect the definition of $\mathfrak{T}$, but only the particular neighborhoods chosen.

The space $\mathfrak{S}^*$ is formed from $\mathfrak{S}$ by identifying "opposite points" (p, v) and $(p, -v)$. The space $\mathfrak{T}^*$ is formed from $\mathfrak{T}$ by the above identification, and also the identification $(p, q) = (q, p)$.

LEMMA 1. *$\mathfrak{T}$ is orientable if and only if M is.*

Let C be a closed curve in $\mathfrak{T} - \mathfrak{S}$ running through (p, q). Let $u_1, \cdots, u_n$ and $v_1, \cdots, v_n$ be independent sets of vectors in M at p and at q respectively. Then keeping q fixed and letting p move in the direction of u_i defines a vector $\bar{u}_i$ in $\mathfrak{T}$; similarly $\bar{v}_i$ is defined, with p fixed. Now the ordered set

$$(\bar{u}_1, \cdots, \bar{u}_n, \bar{v}_1, \cdots, \bar{v}_n) \tag{3.1}$$

defines an orientation of $\mathfrak{T}$ near (p, q). As we follow around C, the points p and q move along closed paths C_1 and C_2 in M.

Suppose first that M is orientable. Then we may carry the vectors of each set around so that they come back to the same vectors, showing that $\mathfrak{T} - \mathfrak{S}$ is orientable.[4] Now take any closed path in $\mathfrak{T}$. Clearly any points on $\mathfrak{S}$ may be pushed away from $\mathfrak{S}$ (replace $\lambda = 0$ by $\lambda > 0$ in $\mathfrak{T}'$). Applying the above proof shows that $\mathfrak{T}$ is orientable.

If M is non-orientable, let C_1 be a closed curve in M reversing the orientation. Then $n > 1$, so that we may choose a point q not on C_1. As p runs around C_1, (p, q) runs arouhd a curve C in $\mathfrak{T}$, which clearly reverses the orientation in $\mathfrak{T}$.

LEMMA 2. *If n is even and M is orientable, $\mathfrak{T}^*$ is orientable; otherwise, $\mathfrak{T}^*$ is non-orientable.*

As before, we need only consider $\mathfrak{T}^* - \mathfrak{S}^*$.

If M is not orientable, the proof above shows that $\mathfrak{T}^*$ is not. Now take M orientable.

Suppose $n > 1$ is odd. Let σ^n be a cell in M, represented with coordinates $x_1, \cdots, x_n$. Let e_i be the unit vector parallel to the x_i-axis in E^n; this maps into a vector $u_i(p)$ at p for each $p \in \sigma$. Let C be a simple closed curve in σ, and let p and q be distinct points of C. Then $(u_1(p), \cdots, u_1(q), \cdots)$ defines an orientation of $\mathfrak{T}^*$ at (p, q). Now move p and q along the two arcs of C into q and p respectively; these vectors are carried into $u_1(q), \cdots, u_1(p), \cdots$ respectively. Thus a permutation of the vectors is defined, formed by n^2 transpositions; n^2 is odd. Since $(q, p) = (p, q)$, we have defined a closed curve in $\mathfrak{T}^*$ reversing the orientation. Note that if n is even, then C preserves the orientation. If $n = 1$, essentially the same proof applies, with $C = M$.

Suppose finally that M is orientable and n is even. Let C^* be any closed curve in $\mathfrak{T}^* - \mathfrak{S}^*$. Take $(p, q) = (q, p)$ in C^*. As we run around C^*, the points p, q separately may be followed in M, and thus (p, q) may be followed in $\mathfrak{T}$. If (p, q) comes back into (p, q) in $\mathfrak{T}$, the proof in the last lemma shows that the orientation is unchanged. If (p, q) comes back into (q, p) in $\mathfrak{T}$, let C be a simple closed curve in M containing p and q, and as in the proof just above, carry p and q into q and p respectively in σ^n, forming a curve D^* in $\mathfrak{T}^*$. We just saw

[4] This follows also at once from the fact that $\mathfrak{T} - \mathfrak{G}$ is part of the cartesian product $M \times M$. We shall need later the vectors defined here.

that C^* followed by D^* does not reverse the orientation, and we noted above that D^* does not either; hence C^* does not.

DEFINITION. If $\mathfrak{T}$ or $\mathfrak{T}^*$ is orientable and M is oriented, the *corresponding orientation* of $\mathfrak{T}$ or $\mathfrak{T}^*$ is defined by the vectors (3.1).

4. The mappings of $\mathfrak{T}$ and $\mathfrak{T}^*$ corresponding to a mapping f of M into an E^m

Take $M \subset \mathrm{E}^{2n+1}$ again. Then a vector u tangent to M at p may be defined as follows by a parametrized curve $P(\lambda)$ starting from p:

$$u = \lim_{\lambda \to 0} \frac{P(\lambda) - p}{\lambda}. \tag{4.1}$$

If ∇f is the mapping of vectors induced by f, than

$$\nabla f(u, p) = \lim_{\lambda \to 0} \frac{f(P(\lambda)) - f(p)}{\lambda}. \tag{4.2}$$

Note that ∇f is a generalization of the gradient of a real-valued function.

Let $| u |$ denote the length of u; $| q - p |$ is the distance from p to q in E^{2n+1}. Define

$$F(p, q) = \frac{f(q) - f(p)}{| q - p |}, \qquad p \neq q, \tag{4.3}$$

$$F(p, u) = \nabla f(u, p) \qquad \text{if } | u | = 1; \tag{4.4}$$

then, taking $P(\lambda)$ as before, if $| u | = 1$ in (4.1),

$$\lim_{\lambda \to 0} F(p, P(\lambda)) = \lim_{\lambda \to 0} \frac{f(P(\lambda)) - f(p)}{| P(\lambda) - p |} = \lim_{\lambda \to 0} \frac{f(P(\lambda) - f(p)}{\lambda}$$
$$= \nabla f(u, p) = F(p, u).$$

Thus it is clear that F is a continuous mapping of $\mathfrak{T}$ into the space V^m of vectors in E^m.

Clearly F defines a mapping F^* of $\mathfrak{T}^*$ into V^{m*}.

LEMMA 3. *If f is regular, then F maps no point of $\mathfrak{S}$ into O, and maps a point (p, q) of $\mathfrak{T} - \mathfrak{S}$ into O if and only if $f(p) = f(q)$.*

This is obvious.

Suppose $f(p) = f(q)$, $p \neq q$, and u is a vector in M at p. Then if $\bar{u}$ in $\mathfrak{T}$ is defined as in §3,

$$\nabla F(\bar{u}, (p, q)) = \lim_{\lambda \to 0} \frac{F(P(\lambda), q) - F(p, q)}{\lambda}$$
$$= \lim_{\lambda \to 0} \frac{1}{\lambda} \left[\frac{f(q) - f(P(\lambda))}{| q - P(\lambda) |} - \frac{f(q) - f(p)}{| q - p |} \right];$$

clearly

$$\nabla F(\bar{u}_i, (p, q)) = -\frac{1}{| q - p |} \nabla f(u_i, p). \tag{4.5}$$

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This is obvious.

Suppose $f(p) = f(q)$, $p \neq q$, and u is a vector in M at p. Then if $\bar{u}$ in $\mathfrak{T}$ is defined as in §3,

$$\nabla F(\bar{u}, (p, q)) = \lim_{\lambda \to 0} \frac{F(P(\lambda), q) - F(p, q)}{\lambda}$$
$$= \lim_{\lambda \to 0} \frac{1}{\lambda} \left[\frac{f(q) - f(P(\lambda))}{| q - P(\lambda) |} - \frac{f(q) - f(p)}{| q - p |} \right];$$

clearly

$$\nabla F(\bar{u}_i, (p, q)) = -\frac{1}{| q - p |} \nabla f(u_i, p). \tag{4.5}$$

Similarly

$$\nabla F(\bar{v}_i, (p, q)) = \frac{1}{|q - p|} \nabla f(v_i, q). \tag{4.6}$$

LEMMA 4. *Let f be a completely regular mapping of M^n into E^{2n}. Then if $f(p) = f(q)$ $(p \neq q)$, F is regular in a neighborhood of (p, q), and maps it over O in V^{2n} with the degree ± 1.*

Since f is completely regular, the vectors

$$\nabla f(u_1, p), \cdots, \nabla f(u_n, p), \qquad \nabla f(v_1, q), \cdots, \nabla f(v_n, q) \tag{4.7}$$

are independent; hence so are the $\nabla F(\bar{u}_i, (p, q))$ and $\nabla F(\bar{v}_i, (p, q))$. The last part of the lemma is a consequence of this.

We consider briefly manifolds with boundary. If f is regular in M and on the boundary ∂M of M (with the obvious definitions, assuming for example that ∂M is a manifold), and $f(p) = f(q)$, $p \neq q$, implies that neither p nor q is in ∂M, we call f *completely regular*. If this is so, F maps no part of $\mathfrak{T}$ coming from points on or near ∂M into O; we need not be explicit as to the definitions of $\mathfrak{T}$ and $\mathfrak{S}$ for such points. Similarly for F^*.

5. Mappings into a vector space with opposite vectors identified

Let $V = V^m$ be the space of vectors in E^m. Let ω be the mapping $\omega(v) = -v$ of V into itself. Form $V^* = V^{m*}$ from V by identifying v with $-v$ for all v; let $\pi(v) = \pi(-v)$ be the corresponding point of V^*. Let O be the zero vector of V, and set $O^* = \pi(O)$.

An arc in $V - O$ from $v \neq O$ to $-v$ maps under π into a closed curve, defining the single element γ not the identity of the fundamental group of $V^* - O^*$.

Let T be an $(m-1)$-plane in V through O; set $\pi(T) = T^*$. For any subdivision K^* of $V^* - O^*$ with no vertex in T^*, let $\mathbf{W}^{*1}$ be the cocycle mod 2 containing those 1-cells cutting T^*; let $\mathbf{W}^{*1}$ be its cohomology class (which is invariantly determined).

DEFINITION. Let f^* map the space R into V^*. If there is a mapping f of R into V such that $f^* = \pi f$, we say f^* is *derivable from a mapping into* V, or is *derivable* simply.

The following lemmas are easily proved.

LEMMA 5. *f^* is derivable if and only if it is derivable over each component of $f^{*-1}(V^* - O^*)$.*

LEMMA 6. *A mapping f^* of a connected complex K into $V^* - O^*$ is derivable if and only if any of the following equivalent conditions are satisfied:*

(α) Each closed curve in K maps into a curve in $V^* - O^*$ defining the identity element of the fundamental group.

(β) Each closed curve in K maps into a curve cutting T^* an even number of times (if this number is finite).

(γ) Each 1-cycle of K maps into a bounding 1-cycle of $V^* - O^*$.

(δ) $f^{*\prime}\mathbf{W}^{*1} = 0$.

DEFINITION. A mapping f^* of a space R into V^* is *locally derivable* (in terms of the f_i below) if there exist open sets $U_1, U_2, \cdots$, covering R and mappings f_i of U_i into V such that:

(a) For each i, $f^* = \pi f_i$ in U_i.

(b) For each i and j, either $f_i = f_j$ or $f_i = \omega f_j$ in $U_i \cap U_j$.

REMARKS. If $\{U_i\}$ and $\{f_i\}$ are replaced by $\{U_i'\}$ and $\{f_i'\}$, the definition is considered as equivalent if (b) holds for the old and new sets together. (See Example 3 below.) Say a set $A \subset R$ is *r-thin* if any singular r-cell in R may be moved away from A by an arbitrarily slight deformation. If $f^{*-1}(O^*)$ is 1-thin, and each non-void $U_i \cap U_j$ is connected, then clearly (b) is a consequence of (a), and any sets of f_i are equivalent. Any mapping of a space into $V^* - O^*$ is clearly locally derivable.

EXAMPLES. The first four illustrate the meaning of local derivability; the fifth illustrates its use in this paper. S and S^* are defined in §6.

(1) Let f_1^* be the identity mapping of $Q^* = S^*$ plus interior into itself. Let f_t map each radius from O^* into itself, shrinking it by the factor t; then $f_0(p) = O^*$ (all p). Then f_0 is locally derivable, while (as proved in Example 9) f_t is not if $t > 0$.

(2) Take f_1 as before, and let f_t map all points of any radius within a distance $1 - t$ of O^* into O^*, extending f_t in an obvious manner over Q^*; again $f_0(p) = O^*$. Then f_1^* is not locally derivable, while f_t is if $t < 1$. However, if a slight change is made for $t > 0$, forming g_t^*, such that each $g_t^{*-1}(O^*)$ is 1-thin in Q^*, then no g_t^* $(t > 0)$ is locally derivable; for the corollary to Lemma 7 below would be contradicted. Hence, by a Remark above, we cannot make (a) hold.

(3) Let $R_1, R_2, \cdots$ be a sequence of disjoint spheres plus interiors in some E^r, approaching a point p_0. Let f map all of E^r but these interiors into O, and map the interiors into $V - O$; set $f^* = \pi f$. Then if $p_0 \in U_i$, there are an infinite number of choices for f_i, making f^* locally derivable in distinct ways.

(4) Let $C_1, C_2, \cdots$ be a sequence of simple closed curves in E^r, with disjoint neighborhoods $U_1', U_2', \cdots$. Map U_i' into V^* so that $f^*(C_i)$ defines γ and so that if $f^*(p) = O^*$ $(p \in E^r - \sum U_i')$, then f^* is continuous. If the C_i converge smoothly to a simple closed curve C, then f^* is locally derivable (but see a remark in Example 2); if the C_i converge to a line segment, it is easily seen that f^* is not locally derivable, even though we can make (a) hold.

(5) Let Q^2 be the unit circle S^1 plus interior in E^2, and let g be the identity mapping of Q^2 into E^2. (We could equally well map Q^r into E^m, $2 \leqq r \leqq m$.) If we identify opposite points of S^1, Q^2 becomes a projective plane P^2; πg becomes a mapping f^* of P^2 into V^*. f^* is locally derivable but not derivable. A diameter of Q^2 becomes a "projective line" L of P^2; set $f^*(L) = L^*$. Subdivide P^2 so that L lies on no vertex a_i, and set $U_i = St(a_i)$ $(St$ = star). If f_i in U_1 is chosen so that $f^* = \pi f_1$ there, then there is a unique choice for each f_i such that if $a_i a_j$ is a simplex in P^2 not cutting L, then $f_i = f_j$ in $St(a_i a_j)$. Now the W^1 of (5.1) below contains just those $a_i a_j$ cutting L; $\mathbf{W}^1 \neq 0$. If we omit a small concentric disc from Q^2, and hence from P^2, a Möbius strip M^2 illustrating Lemma 6 is formed, with $f'\mathbf{W}^{*1}$ being the cohomology class of the part of the above W^1 lying in M^2.

(6) Take Q^2, S^1, as in (5). Map S^1 into a curve in $V^* - O^*$ so as to define γ; map radii of Q^2 into segments from O^*. Then f^* is not locally derivable; in the notations of Lemma 8 below,

$$\delta f^{*\prime} W^1 \cdot Q^2 = f^{*\prime} W^1 \cdot \partial Q^2 = W^1 \cdot f^* S^1 = 1_2 .$$

If f^* is any mapping of Q_2 into $V^* - O^*$ with an inner point p going into O^*, then a slight alteration of f^* may be made which will give a mapping g^* for which a neighborhood of p is mapped as above; then g^* is not locally derivable.

DEFINITION. Let f^* be a locally derivable mapping of the complex K into V^* such that $f^{*-1}(O^*)$is 0-thin, i.e. nowhere dense in K. Then the cohomology class W^1 of f^* is defined as follows. Take the U_i and f_i as before. Let K_1 be a subdivision of K so fine[5] that each star $St_1(a_i)$ of a vertex a_i of K_1 is in some U_{λ_i}. Set

$$W^1(a_i a_j) = \begin{cases} 0_2 & \text{if } f_{\lambda_i} = f_{\lambda_j} \text{ in } St_1(a_i a_j), \\ 1_2 & \text{if } f_{\lambda_i} = wf_{\lambda_j} \text{ in } St(a_i a_j). \end{cases} \tag{5.1}$$

Then $\mathbf{W}^1$ is the cohomology class of $W^1 = \sum W^1(a_i a_j) a_i a_j$.

LEMMA 7. *W^1 is well defined, and is a cocycle (coefficients* mod **2**). *Altering the f_i alters it by a coboundary. Its class $\mathbf{W}^1$ is independent of the choice of K, the U_i and the f_i, provided that the new f_i are equivalent to the old. If $f^{*-1}(O^*)$ is 1-thin, then f^* is derivable if and only if $\mathbf{W}^1 = 0$.*

REMARKS. Any such W^1 is called a *characteristic cocycle* of f^*. By standard theory, that $\mathbf{W}^1$ is independent of K is a consequence of the following statement: If K_1 and K_2 are such that each $St_2(a_{2i})$ lies in some $St_1(a_{1\nu_i})$, so that a simplicial mapping $\phi(a_{2i}) = a_{1\nu_i}$ of K_2 into K_1 is defined, and W_1^1 is a characteristic cocyle of K_1, then $W_2^1 = \phi'\, W_1^1$ (ϕ' = dual of ϕ) is a characteristic cocycle of K_2; by definition, $W_2^1(a_{2i}a_{2j}) = W_1^1(\phi(a_{2i}a_{2j})) = W_1^1(a_{1\nu_i}a_{1\nu_j})$.

First, since $f^{*-1}(O^*)$ is nowhere dense in K_1, the definition of W^1 is obviously unique. Next, suppose a definite f_k is changed to ωf_k. Then if $i_1, \cdots, i_r$ are the i such that $\lambda_{i_s} = k$, $W^1(a_i a_j)$ is changed if and only if just one of i, j lies among the i_s; thus, if δ denotes coboundary,

$$\text{new } W^1 = \text{old } W^1 + \delta \sum (a_{i_s})_2 .$$

Now for any sets $\{U_i\}$, $\{U_i'\}$ and equivalent $\{f_i\}$, $\{f_i'\}$, choose K_1 so that $St_1(a_i) \subset U_{\lambda_i} \cap U'_{\mu_i}$. Let f_{ij} and f'_{ij} denote f_{λ_i} and f'_{μ_i} respectively, considered in $U_{\lambda_i} \cap U'_{\mu_j}$ (if this is not void); all the f are equivalent. Clearly the f_{λ_i} and the f_{ij}, also the f'_{μ_i} and f'_{ij}, define the same W^1. The proof above shows that changing the f_{ij} to the f'_{ij} alters W^1 by a coboundary.

Now suppose K_1 and K_2 are as described above. In using (5.1) to define W_2^1, since $St_2(a_{2i}) \subset St_1(a_{1\nu_i})$, we may take for the f_k defined about a_{2i} the same as the f_k defined about $a_{1\nu_i}$; then clearly $W_2^1(a_{2i}a_{2j}) = W_1^1(a_{1\nu_i}a_{1\nu_j})$.

[5] From the corollary below it is clear that the f_i may be defined in the stars in K, so that no subdivision is necessary in reality.

Suppose $f^* = \pi f$. Since $f^{*-1}(O^*)$ is 1-thin, the given set $\{f_i\}$ is equivalent to the single mapping f; hence W^1 may be defined using f. Since $W^1 = 0$, $\mathbf{W}^1 = 0$.

Suppose finally that $\mathbf{W}^1 = 0$. For a given subdivision K_1, and sets $\{U_i\}$, $\{f_i\}$, we have $W^1 \doteq \delta X^0$ for some $X^0 = \sum \alpha_i a_i$ ($\alpha_i = 0_2$ or 1_2). If $St_1(a_i) \subset U_{\lambda_i}$, set $f_i' = f_{\lambda_i}$ in $St_1(a_i)$. For each i with $\alpha_i = 1_2$, alter f_i' to $\omega f_i'$; then (see a proof above) the new $W^1 = 0$. Now for any p in K, say $p \in St(a_i)$; set $f(p) = f_i'(p)$. This makes f continuous in K, and $f^* = \pi f$.

EXAMPLE (7). Let f map a circle C into V so that just one point p_0 goes into O, and set $f^* = \pi f$. Let U_1 and U_2 cover C, p not being in U_2. Then f_1 and f_2 may be chosen so that W^1 is not $\backsim 0$, though f^* is derivable. Note that $f^{*-1}(O^*) = p_0$, which is not 1-thin in C.

COROLLARY. *Any locally derivable mapping f^* of a cell (in fact, of a complex with vanishing* 1*-cohomology group* mod 2*) is derivable, if $f^{*-1}(O^*)$ is* 1*-thin in K.*

A direct proof for a cell may be given simply if we build it up as in (b) of Lemma 12.

We shall give a lemma which shows immediately whether or not a mapping is locally derivable or derivable, provided that $f^{*-1}(O^*)$ is 1-thin. Since it involves notions[6] not yet generally known, and the lemma is not needed here, we shall be rather brief.

A geometric r-G-cochain $X = X^r$ in a point set R is a pair $\bar{X} \supset \bar{X}'$ (nucleus and nuclear boundary) of closed sets, $(r-1)$-thin and r-thin respectively, and a group of homomorphisms $X \cdot A^r$ of those integral singular chains A^r for which $A \cap \bar{X}' = \partial A \cap \bar{X} = 0$ into G, with the properties $X \cdot A = 0$ if $A \cap \bar{X} = 0$, $X \cdot \partial B^{r+1} = 0$ if $B \cap \bar{X}' = 0$. If G and H are paired to J, set $X \cdot \sum h_i \sigma_i^r = \sum (X \cdot \sigma_i^r) \cdot h_i$. The coboundary δX of X is the pair of sets $\bar{X}', 0$, with the definition $(\delta X) \cdot A^{r+1} = X \cdot \partial A^{r+1}$. To any cochain X^r in a complex corresponds in a fairly obvious fashion a geometric cochain with nucleus and nuclear boundary formed from duals of r- and $(r+1)$-cells.

Since T^* and O^* are 0-thin and 1-thin in V^* respectively, they give a geometric cochain W^1 of V^*, such that $W^1 \cdot \sigma^1$ counts the number of times mod 2 that σ^1 crosses T^*. This is the geometric analogue of the W^1 first defined. If f^* is as in the lemma below, then

$$W^1_{f^*} \cdot A^1 = W^1 \cdot f^* A^1$$

defines a geometric cocycle $W^1_{f^*} = f^{*\prime} W^1$ in K. Note that $\delta f^{*\prime} W^1 = f^{*\prime} \delta W^1$.

LEMMA 8. *Let f^* be any mapping of a complex K into V^* such that $f^{*-1}(T^*)$ and $f^{*-1}(O^*)$ are* 0*-thin and* 1*-thin respectively. Then f^* is locally derivable if and only if $\delta f^{*\prime} W^1 = 0$, and is derivable if and only if $f^{*\prime} W^1 \backsim 0$.*

REMARK. Using ordinary cocyles, we may replace the condition $\delta f^{*\prime} W^1 = 0$ by the following: For any arbitrarily fine subdivision K_1 of K there is an arbitrarily small deformation of f^* into f_1^* such that $f_1^{*\prime} W^1$ is defined and is a cocycle.

Suppose f^* is locally derivable, using $\{U_i\}$ and $\{f_i\}$. Take any singular chain A^2, with $\partial A^2 \cap f^{*-1}(O^*) = 0$, so that $\delta W^1_{f^*} \cdot A$ is defined. Using a fine

[6] See H. Whitney, *Geometric methods in combinatorial topology*, not yet published.

subdivision of A, we may write $A = \sum \alpha_i \sigma_i^r$, each σ_i^r in some U_{λ_i}; since $f^{*-1}(O^*)$ is 1-thin, we may deform A slightly so that $\sum \alpha_i (\delta W_{f^*}^1 \cdot \sigma_i^r)$ is defined; it equals $\delta W_{f^*}^1 \cdot A$. Using f_{λ_i}, it is clear that $\delta W_{f^*}^1 \cdot \sigma_i^r = W^1 \cdot \partial f^* \sigma_i^r = O_2$; hence $\delta W_{f^*}^1 = 0$.

Suppose $\delta W_{f^*}^1 = 0$. Each point $p \in K$ is in a connected neighborhood U such that any 1-cycle in U bounds. Choose $p_0 \in U$ so that $f^*(p_0) \neq O^*$; choose $f(p_0)$ so that $\pi f(p_0) = f^*(p_0)$. For any $p' \in U$ with $f^*(p') \neq O^*$, choose an arc C in U from p_0 to p' with $f^*(C) \subset V^* - O^*$ (which is possible, since $f^{*-1}(O^*)$ is 1-thin); $f(p')$ is uniquely definable so that $f(p'')$ exists and is continuous in C, and $f^* = \pi f$ there. We must show that going around a closed curve A^1 in U and through p_0 ($f^*(A^1) \subset V^* - O^*$) brings $f(p_0)$ back to its original value. Say $A^1 = \partial B^2$. Then $W_{f^*}^1 \cdot A^1 = \delta W_{f^*}^1 \cdot B^2 = 0$, from which the statement clearly follows. Thus we prove (a); since $f^{*-1}(O^*)$ is 1-thin, (b) follows also for suitable $\{U_i\}$.

Suppose f^* is derivable; say $f^* = \pi f$. Then $\mathbf{W}_{f^*}^1 = f' \pi' \mathbf{W}^1 = f' 0 = 0$, since the cohomology groups of V vanish.

Suppose finally $W_{f^*}^1 = \delta W^0$. Then for any closed curve A^1 in K with $f^*(A^1) \subset V^* - O^*$, $W_{f^*}^1 \cdot A^1 = W^0 \cdot \partial A^1 = 0$; it follows by Lemma 5 and Lemma 6 (β) that f is definable throughout K.

LEMMA 9. *Let f_0^* be a locally derivable mapping of the complex K into V^*, with $f^{*-1}(O^*)$ nowhere dense in K. Then there is an arbitrarily slight deformation f_t^* of f_0^* ($0 \leqq t \leqq 1$) such that $F^*(t, p) = f_t(p)$ is locally derivable, with mappings F_i equivalent in $0 \times K$ to the given $f_{0,i}$ there, and such that $f_1^*(K^{2n-1}) \subset V^* - O^*$. If A is a closed subset of K with $f^*(A) \subset V^* - O^*$, we may make $f_t^* = f_0^*$ in A.*

REMARK. As a consequence, it is easily seen (using the various derived of K) that $f_1^{*-1}(O^*)$ may be made $(2n-1)$-thin in K.

We may suppose $f_{0,i}$ is defined in $St(a_i)$; we shall define F_i in $I \times St(a_i)$. Let $St'(a_i)$ be the barycentric star of a_i, containing all points on cells of the first derived of K which have a_i as a vertex. Then $\overline{St}'(a_i) \subset St(a_i)$ and $\sum \overline{St}'(a_i) = K$. Take a fixed i. We may define $f_{t,i}$ in $\overline{St}(a_i)$ for $0 < t \leqq t_1$ so that

$$f_{t_1,i}(K^{2n-1} \cap \overline{St}'(a_i)) \subset V^* - O^*,$$

$$f_{t,i}(p) = f_{0,i}(p) \qquad (p \in \partial St(a_i) \cup (St(a_i) \cap A)),$$

with $f_{t,i}(p)$ arbitrarily close to $f_{0,i}(p)$. For each j such that $a_i a_j$ is a 1-cell of K, set

$$f_{t,j}(p) = f_{t,i}(p) \text{ or } \omega f_{t,i}(p) \qquad (p \in St(a_i a_j)),$$

according as which holds for $t = 0$, and set $f_{t,j}(p) = f_{0,j}(p)$ elsewhere. This clearly defines $F^*(t, p)$ for $p \in K$ and $0 < t \leqq t_1$. Do this for each a_i in turn, using successive subintervals of (0, 1). (K may be infinite, if it is "locally finite.") If the successive alterations are small enough, we will have automatically $F^*(t, p) \subset V^* - O^*$ ($p \in K^{2n-1} \cap \overline{St}'(a_i)$) for previous i. Thus all of F^* is defined.

6. Locally derivable mappings into V^{2n*}

DEFINITIONS. Let $S = S^{2n-1}$ be the unit sphere in V^{2n}, and set $S^* = S^{2n-1*} = \pi S$. Then S^* is a projective space, which is orientable, since $2n - 1$ is odd. We may suppose the spaces oriented so that, combinatorially, $\pi V = 2V^*$ and $\pi S = 2S^*$; thus π has the degree 2. Let P and P^* be the projections of $V - O$ and $V^* - O^*$ along rays from O and O^* onto S and S^* respectively.

Let L^* be a ray from O^* in V^*. Given any subdivision K^* of $V^* - O^*$ with $(K^*)^{2n-2} \subset V^* - L^*$, we may define a cocycle $W = W^{2n-1}$ in K^* as follows. For any $\sigma = \sigma^{2n-2}$, $W^{2n-1}(\sigma) = KI(\sigma, L^*)$. This is the algebraic number of times that $P^*\sigma$ covers $L^* \cap S^*$ in S^* or, that L^* cuts through σ.

Since $L^* - O^*$ is $(2n-2)$-thin in $V^* - O^*$, we can consider W as a geometric cocycle in $V^* - O^*$. Since O^* is not $(2n-1)$-thin in V^* for $n \geqq 2$, (in fact, not 2-thin; compare Example (6) and the corollary to Lemma 7), we cannot properly consider it as a cochain in V^*, whose coboundary has the nucleus O^*. In fact, δW^* may be considered in a general or in a restricted way:

(A) If a singular chain $A^{2n} \subset V^*$ has $\partial A^{2n} \subset V^* - O^*$, we may set $\delta W \cdot A^{2n} = W \cdot \partial A^{2n}$. For example, if $A^{2n} = S^*$ plus interior, then $\partial A = S^*$, and $\delta W \cdot A = 1$.

(B) If we define $\delta W \cdot A^{2n}$ only for singular chains A which may be expressed as $A = \sum \alpha_i \sigma_i^{2n}$, with $\partial \sigma_i^{2n} \subset V^* - O^*$, then we have always $\delta W \cdot A = W \cdot \partial A \equiv 0 \pmod 2$; see the next lemma.

DEFINITIONS. For any singular cycle $A^{2n-1} \subset V^* - O^*$, the *degree* $d(A)$ of A^{2n-1} is $W^{2n-1} \cdot A^{2n-1}$. If f^* maps a cycle A^{2n-1} into $V^* - O^*$, the *degree* $d_{f^*}(A)$ *of* f^* *in* A *about* O^* is $d(f^*A)$. This is clearly invariant under deformations; furthermore, combinatorially,

$$P^*f^*A = \alpha S^*, \qquad \alpha = d_{f^*}(A). \tag{6.1}$$

For any singular chain A^{2n} with $\partial A^{2n} \subset V^* - O^*$, the *degree* $d(A^{2n})$ *of* A^{2n} *over* O^* is $W^{2n-1} \cdot \partial A^{2n}$ $(= \delta W \cdot A)$; define $d_{f^*}(A^{2n})$ similarly. These are obvious generalizations of the degree about O or over O in V.

LEMMA 10. *Suppose* $n \geqq 2$. *If* f^* *maps a complex* K *into* V^* *so that* $f^*K^{2n-1} \subset V^* - O^*$, *then* $\delta f^{*\prime} W^{2n-1} \equiv 0 \pmod 2$, *so that*

$$\tilde{W}_{f^*}^{2n} = \tfrac{1}{2}\delta f^{*\prime} W^{2n-1} \quad \textit{exists.} \tag{6.2}$$

The strict meaning of $\tilde{W}_{f^*}^{2n}$ is given by its action on any $2n$-G-chain $A^{2n} = \sum g_i \sigma_i^{2n}$, which is as follows:

$$\tilde{W}_{f^*}^{2n} \cdot \sum g_i \sigma_i^{2n} = \sum g_i (\tilde{W}_{f^*}^{2n} \cdot \sigma_i^{2n}), \tag{6.3}$$

$$\tilde{W}_{f^*}^{2n} \cdot \sigma = \tfrac{1}{2}\delta f^{*\prime} W^{2n-1} \cdot \sigma = \tfrac{1}{2}(W^{2n-1} \cdot f^* \partial \sigma). \tag{6.4}$$

Since $2n - 1 > 1$, $\partial\sigma$ is simply connected; hence there is a mapping f of $\partial\sigma$ into $V - O$ such that $f^* = \pi f$. Say $Pf\partial\sigma = \alpha S$. Then

$$P^*f^*\partial\sigma = P^*\pi f \partial\sigma = \pi P f \partial\sigma = 2\alpha S^*;$$

hence

$$\delta f^{*\prime} W^{2n-1} \cdot \sigma = W^{2n-1} \cdot f^* \partial\sigma = W^{2n-1} \cdot P^* f^* \partial\sigma = 2\alpha,$$

from which the lemma follows.

REMARKS. By Lemma 9 and the remark following it, a slight deformation of a locally derivable f^* will give an f_1^* such that $f_1^{*\prime} W^{2n-1}$ and $\tilde{W}^{2n}_{f_1^*}$ are geometric cochains.

It is not true that $\partial A^{2n} = 0$ implies $\tilde{W}^{2n}_{f^*} \cdot A = \frac{1}{2}(\delta f^{*\prime} W^{2n-1} \cdot A) = \frac{1}{2}(W^{2n-1} \cdot f^* \partial A) = 0$; see Example (8) below. The second expression has in general no meaning, say if the coefficients are integers mod 2. But it holds for integral A, as is apparent from (6.4).

DEFINITION. Given a locally derivable mapping f^* of K into V^*, and a $2n$-G-chain A in K such that $f^* \partial A \subset V^* - O^*$, make a slight deformation of f^* into f_1^* such that $f_1^*(K^{2n-1}) \subset V^* - O^*$, as in Lemma 9, keeping ∂A^* in $V^* - O^*$; let the *divided degree* of A under f be

$$\tilde{d}_{f^*}(A) = \tilde{W}^{2n}_{f_1^*} \cdot A \qquad \text{(which is an element of } G\text{)}. \tag{6.5}$$

Note that this defines $\tilde{d}_{f^*}(A)$ for locally derivable singular chains A with $\partial A \subset V^* - O^*$. It need not vanish for cycles (see Example (8) below).

LEMMA 11. *The above degree is independent of the choice of f_1^*, vanishes for chains mapped into $V^* - O^*$, and is invariant under deformations f_t^* such that $F^*(t, p) = f_t^*(p)$ is locally derivable in $I \times K$ and $f_t^*(\partial A) \subset V^* - O^*$. It vanishes for all boundaries ∂B^{2n+1}, in fact, for all "cycles of the first kind," expressible as $\sum g_i A_i^{2n}$ ($g_i \subset G$, A_i^{2n} integral cycles).*

We prove the last part first. Supposing $f_1^*(K^{2n-1}) \subset V^* - O^*$, we have

$$\tilde{W}^{2n}_{f_1^*} \cdot \sum g_i A_i^{2n} = \sum g_i (\tilde{W}^{2n}_{f_1^*} \cdot A_i^{2n}) = \sum g_i [\tfrac{1}{2}(W^{2n-1} \cdot f^* \partial A_i^{2n})] = 0;$$

also, if $A^{2n} = \partial B^{2n+1}$, say $B^{2n+1} = \sum g_i \sigma_i^{2n+1}$, then $A^{2n} = \sum g_i \partial \sigma_i^{2n+1}$ is a cycle of the first kind, so that $\tilde{W}^{2n}_{f_1^*} \cdot A^{2n} = 0$.

To prove the rest, it is sufficient to show that if

$$f_t^* \partial A \cup f_0^*(K^{2n-1}) \cup f_1^*(K^{2n-1}) \subset V^* - O^*,$$

and $F^*(t, p) = f_t^*(p)$ is locally derivable, then

$$\tilde{W}^{2n}_{f_1^*} \cdot A = \tilde{W}^{2n}_{f_0^*} \cdot A.$$

By a slight deformation of F^*, keeping $F^*(0, p)$ and $F^*(1, p)$ fixed, we may suppose that F^* maps $(I \times K)^{2n-1}$ into $V^* - O^*$, and is locally derivable (see Lemma 9); suppose this is done. The chain A determines chains $0 \times A$, $1 \times A$, $I \times A$ in $I \times K$ (if we use the usual subdivision of $I \times K$). Using the fact just proved, we have $\tilde{W}^{2n}_{F^*} \cdot [\partial(I \times A)] = 0$; hence, since $\partial(I \times A) = 1 \times A - 0 \times A - I \times \partial A$ and $F^*(I \times \partial A) \subset V^* - O^*$, $\tilde{W}^{2n}_{F^*} \cdot (1 \times A) = W^{2n}_{F^*} \cdot (0 \times A)$, which give the result.

MORE EXAMPLES. (8) (Compare Examples (1) and (5).) Let Q^{2n} be S^{2n-1} plus interior R^{2n}, and form P^{2n} by identifying opposite points of S^{2n-1}. Thus

$P^{2n} = S^*$ plus R^{2n}. Set $\tilde{f}$ = identity in Q^{2n}, and $\tilde{f}^* = \pi\tilde{f}$; this gives a mapping f^* of P^{2n} into V^*. Now P^{2n} is a projective space, of even dimension $2n$, and hence is non-orientable; it has a fundamental $2n$-cycle A^{2n} mod 2. If we subdivide it so that O is interior to a $2n$-cell σ_0^{2n} of Q^{2n}, it is clear that

$$\tilde{W}_{f^*}^{2n} \cdot A^{2n} = [\tfrac{1}{2}(W^{2n-1} \cdot \pi\partial\sigma_0^{2n})]_2 = 1_2 .$$

Of course A^{2n} is not a cycle of the first kind. In the case $n = 1$, this type of mapping illustrates Theorem 1; see the discussion there for $n = 1$.

We may deform f^* by contracting $f^*(K)$ to O^*, and then pulling along a single ray away from O^*. This shows that the assumption in Lemma 11 that F^* is locally derivable cannot be omitted.

(9) Let f^* be the identity mapping of S^* into itself. We may consider S^* as an integral cycle; $W^{2n-1} \cdot S^* = 1$. Of course S^* bounds its interior (combinatorially); but by Lemma 11, there exists no complex K containing S^* in which S^* bounds, such that there is a locally derivable mapping of K into V^* which equals the identity in S^*.

7. The intersection number of a mapping

DEFINITIONS. Let f be a completely regular mapping of M^n (with or without boundary) into E^{2n}; we suppose E^{2n} oriented. Suppose first that M is orientable and n is even. Choose an orientation of M. Suppose $f(p) = f(q)$; let $u_1, \cdots, u_n$ and $v_1, \cdots, v_n$ be sets of vectors in M as in §3, each set determining the positive orientation of M. Then we say this self-intersection is of *positive* or *negative* type according as the vectors (4.7) determine the positive or negative orientation of E^{2n}. Also, by (4.5) and (4.6) (see the end of §3), the type is positive or negative according as the degree in Lemma 4 is 1 or -1. Note that this is independent of the orientation chosen for M. The *intersection number* I_f of f is the algebraic number of self-intersections in this case, and the number of self-intersections mod 2 if M is non-orientable or n is odd. In the case $n = 1$, it is possible to determine I_f in a certain sense as an integer; see the discussion following Theorem 1.

Define F and F^* as in §4. Clearly F^* is locally derivable. Considering $\mathfrak{T}^*$ as a singular chain, with integers or integers mod 2 as coefficients according as it is orientable or not, define the degree $d_{F^*}(\mathfrak{T})$ and divided degree $\tilde{d}_{F^*}(\mathfrak{T}^*)$ as in §6. Let D denote the distant intersection of M (see [3], §18).

THEOREM 1. *Let f be a completely regular mapping of M^n into E^{2n} (M may have a boundary). Let the number of self-intersections be finite. Then*

(a) *If M is orientable and n is even,*

$$I_f = \tilde{d}_{F^*}(\mathfrak{T}^*) = \tfrac{1}{2}d_{F^*}(\mathfrak{T}) = \tfrac{1}{2}(D \cdot M^n) \qquad (\textit{an integer}). \tag{7.1}$$

(b) *If M is non-orientable or n is odd,*

$$I_f = \tilde{d}_{F^*}(\mathfrak{T}^*) \qquad (\textit{an integer} \bmod 2). \tag{7.2}$$

(c) *If M is orientable and n is odd,*

$$d_F(\mathfrak{T}) = D\cdot M^n = 0. \tag{7.3}$$

The theorem shows that F^* has more interest than F in this connection. We discuss the case $n = 1$ further later.

Suppose first that M is orientable and n is even. Take a self-intersection $f(p) = f(q)$, $p \neq q$. The points of $\mathfrak{T}$ near (p, q) are mapped by F over O with the degree 1 or -1, according as the intersection is of positive or negative type. The same is true for the points of $\mathfrak{T}$ near (q, p). Thus each positive [negative] self-intersection contributes 2 [-2] to d_F, and $I_f = \frac{1}{2}d_F$, which is part of (7.1). That $I_f = \frac{1}{2}(D\cdot M)$ follows easily from the definition of D.

Next we prove (c). Take (p, q) again, and define the vectors u_i and v_i as before. Neighborhoods of (p, q) and (q, p) are oriented by $\bar{u}_1, \cdots, \bar{v}_n$ and $\bar{v}_1, \cdots, \bar{u}_n$ respectively; since n is odd, they are mapped over O in V^{2n} with opposite degrees; hence they contribute 0 to d_F. Thus $d_F = 0$. Similarly $D\cdot M = 0$.

Now consider F^* in $\mathfrak{T}^*$, $\mathfrak{T}^*$ orientable; clearly F^* is locally derivable. The only points mapped into O^* by F^* are points $(p, q) = (q, p)$, $f(p) = f(q)$, $p \neq q$. Let U be a neighborhood of (p, q) in $\mathfrak{T}$; it corresponds to a neighborhood U^* in $\mathfrak{T}^*$, say under the mapping ϕ. Then $F^* = \pi F\phi^{-1}$ in U^*. Let $\sigma^* = \sigma^{*2n}$ be a cell in U^* about (p, q), oriented like $\mathfrak{T}^*$. If $\sigma = \phi^{-1}\sigma^*$, then as noted above, F maps σ over O with the degree 1 or -1, according as the self-intersection is positive or negative. Since π is of degree 2, πF maps σ over O^* with the degree 2 or -2, i.e.

$$W^{2n-1}\cdot \pi F\partial\sigma \;=\; W^{2n-1}\cdot F^*\partial\sigma^* \;=\; 2 \text{ or } -2.$$

Hence, by (6.4),

$$\tilde{W}^{2n}_{F^*}\cdot\sigma^* \;=\; \tfrac{1}{2}(W^{2n-1}\cdot F^*\partial\sigma^*) \;=\; 1 \text{ or } -1;$$

using (6.5), we see that σ^* contributes 1 or -1 to $\tilde{d}_{F^*}$. Thus $I_f = \tilde{d}_{F^*}$ (M orientable and n even), proving the rest of (7.1).

Finally we prove (b). In this case, $\mathfrak{T}^*$ is non-orientable, so that it is considered as a chain mod 2 in forming $\tilde{d}_{F^*}(\mathfrak{T}^*)$. As before, $\tilde{W}^{2n}_{F^*}\cdot\sigma^* = 1$ or -1, so that (using (6.3)) $\tilde{W}^{2n}_{F^*}\cdot(1_2\sigma^*) = 1_2$; clearly (7.2) follows.

THE CASE $n = 1$. Though $\mathfrak{T}^*$ is non-orientable, we shall use integer coefficients. For M we may take the unit circle in the plane E^2, described by an angle θ ($0 \leqq \theta < 2\pi$). $\mathfrak{T}^*$ is a Möbius strip with boundary curve $\mathfrak{S}^*$. Any point of $\mathfrak{T}^* - \mathfrak{S}^*$ is a pair (θ, θ'); we may take $\theta < \theta'$. We may let (θ, θ) denote the points of $\mathfrak{S}^*$. If we express $\mathfrak{T}^*$ as the image of the triangle (x, y), $0 \leqq x \leqq y \leqq 1$, the vertical and horizontal sides will each map into the set $\mathfrak{S}'$ of points $(0, \theta)$. Let $\mathfrak{T}^*_1$ be an integral chain covering $\mathfrak{T}^*$; then, combinatorially, $\partial\mathfrak{T}^*_1 = \mathfrak{S}^* + 2\mathfrak{S}'$, so that by (6.3) and (6.4), with $W^1 = W^{2n-1}$ as in §6, using (7.1) to define I_f as an integer,

$$I_f = \tilde{W}^2_{F^*}\cdot\mathfrak{T}^*_1 = \tfrac{1}{2}(W^1\cdot F^*\partial\mathfrak{T}^*_1) = \tfrac{1}{2}(W^1\cdot F^*\mathfrak{S}^*) + W^1\cdot F^*\mathfrak{S}'.$$

$W^1 \cdot A^1$ is the degree of A^1 about O^* in V^{2*}; clearly $\frac{1}{2}(W^1 \cdot F^* \mathfrak{S}^*)$ is the number of times $F\mathfrak{S}^*$ (or more properly, ∇F^v, $v \in \mathfrak{S}$) winds about O in V^2, i.e. is the "rotation number" of f in M. Now if the point $\theta = 0$ is chosen so that $f(0)$ is on a line L of support to $f(M)$ in E^2 (i.e. $f(M)$ lies on one side of L), it is easy to see that $F\mathfrak{S}'$ makes half a turn about O, so that $F^*\mathfrak{S}'$ makes one turn about O^*, and $W^1 \cdot F^*\mathfrak{S}' = \pm 1$. Hence I_f always differs by unity from the rotation number of $f(M)$, again illustrating the warning in §6.

THEOREM 2. *Let M^n be compact (closed, or compact and with boundary). Then the intersection number I_f is invariant under regular deformations; if M has a boundary, we assume that during the deformation, the boundary is not carried across M:*

$$F_t(\partial M) \cap f_t(M - \partial M) = 0 \qquad \text{(all } t\text{)}. \tag{7.4}$$

This means $I_{f_1} = I_{f_0}$, f_0 and f_1 being completely regular; other f_t need only be regular.

REMARKS. If we omit (7.4), it would be simple to give a formula expressing the change in I_f in terms of the crossings of ∂M through M. Thus if M^1 is a closed arc, mapped into the written letter e, $I_f = \pm 1$; if we pull it out into a line segment, $I_f = 0$, and ∂M^1 has crossed M^1 once.

If M is not compact (and no further conditions are assumed), the theorem fails. For example, let M^1 be the open segment $0 < \lambda < 4$. For $0 \leqq t \leqq 1$, let f_t be the mapping given by $(x, y) = f_t(\lambda)$:

$$x = \lambda, \qquad y = t/\lambda \qquad (0 < \lambda < 2),$$

and let the rest of M^1 be carried counter-clockwise around onto the line $y = 1$, and back across the y-axis along $y = 1$. Then $I_{f_t} = -1$ if $t > 0$, but $I_{f_0} = 0$.

Of course I_f is not invariant under deformations not assumed regular.

Suppose first that M is a manifold (without boundary). Since each F_t^* maps $\mathfrak{S}^* = \partial\mathfrak{T}^*$ into $V^* - O^*$, and is continuous, $\tilde{d}_{F_1^*} = \tilde{d}_{F_0^*}$, by Lemma 11. By the last theorem, $I_{f_0} = I_{f_1}$.

Now suppose M has a boundary. Then $\partial\mathfrak{T}^*$ consists of points of $\mathfrak{S}^*$ and points (p, q), $p \in \partial M$ or $q \in \partial M$. The hypothesis shows again that F_t leaves $\partial\mathfrak{T}^*$ in $V^* - O^*$, and again $I_{f_0} = I_{f_1}$.

II. THE IMBEDDING THEOREM

8. Discussion of the theorem

We first give theorems on the possibility of altering I_f, and of deforming f (keeping I_f fixed) so as to reduce the number of self-intersections. The imbedding theorem follows from these.

THEOREM 3. *Given any compact M (with or without boundary), there is a proper completely regular mapping f of M^n into E^{2n} with any desired I_f.*

Let f_0 be a completely regular mapping of M^n into E^{2n} (see [1], Theorem 3; it is not hard to extend this theorem to the case that M has a boundary, for example if ∂M is a manifold). Now take a small piece of M, flatten it, cut it

out, and replace it by a piece with a single self-intersection, defined in §2, or by this piece altered by a reflection in E^{2n}. Then I_{f_0} is changed by 1 or -1 at will. Thus I_f may be increased or decreased by 1, and hence may be brought to any desired value.

THEOREM 4. *Let n be $\geqq 3$, and let f_0 be a proper regular mapping of a closed M^n into E^{2n}, with at most a finite number of self-intersections. Then there is a regular deformation f_t of f into a proper completely regular mapping f_1, such that the number of self-intersections under f_1 is increased by 2. If the number is $> |I_f|$, or is >0 if n is odd or M is non-orientable, we may decrease the number by 2. If M has a boundary, we may keep ∂M fixed, and keep M from cutting through ∂M.*

PROBLEM. Does the theorem hold for $n = 2$?

For the moment we shall prove merely that the number of self-intersections may be increased by 2. The rest of the proof will occupy §§9 through 12.

First, f may be made completely regular, as in [1]. Suppose this is done. Take a small nearly flat portion σ^n of $f(M)$. Take $p \in \sigma^n$, and pull p out away from σ^n, around and towards another point q of σ^n, pulling a small portion of σ^n along with it so we always have a regular mapping. We may make the moving part of M avoid any other points of M or ∂M. Say σ^n was approximately in the $(x_1, \cdots, x_n)$-plane in E^{2n}, and say q is the origin O, and we are pulling p around in the (x_1, x_{n+1})-plane, and down along the x_{n+1}-axis. We may tip the moving portion of σ^n near p to be parallel to the $(x_1, x_{n+2}, \cdots, x_{2n})$-axis, as we move p. Now pull p through O. Then if the mapping f_1 is obtained, the new self-intersections under f_1 are clearly on the x_1-axis T. The part of $f_1(\sigma^n)$ near p intersects T at points on each side of O. These latter points are new self-intersections, as required.

The rest of the proof of Theorem 4 runs as follows. We need merely show that two self-intersections of opposite types (or any two, if M is non-orientable or n is odd) can be gotten rid of by a regular deformation. Say

$$f(p_1) = f(p_2) = q, \qquad f(p_1') = f(p_2') = q',$$

these being of opposite types. Let C_1 and C_2 be non-intersecting curves in M, C_i joining p_i and p_i', neither passing through any other point where f has a self-intersection. Then $B_i = f(C_i)$ joins q to q', and $B = B_1 \cup B_2$ is a simple closed curve in $f(M)$. We let B bound a smooth 2-cell σ, which touches M only at B, and show that the part of M near C_2 may be deformed along near σ so that it passes beyond B_1, thus removing the two self-intersections. Until §12 we assume that M is orientable and n is even.

THEOREM 5. *Any smooth n-manifold may be imbedded in E^{2n}.*

REMARK. This theorem may be generalized to the case of manifolds with boundary, but we shall not discuss in what generality. For example, if ∂M is a manifold, the extension is not difficult. For the proof, see the proof of Case IV of Theorem 6 in the following paper.

For $n = 1$, the theorem is trivial. For $n = 2$, we imbed the sphere, projective plane, or Klein bottle, in E^4, and add the necessary number of handles to obtain the given manifold. (An imbedding of the projective plane is easily found from [3] p. 108.)

Now we suppose that $n \geqq 3$. First, by [1], Theorem 3, there is a proper completely regular mapping f_0 of M^n into E^{2n}. Suppose M is closed. Then, by Theorem 3, we alter f_0 to f_1 so that $I_{f_1} = 0$, and by Theorem 4, we may get rid of all of the intersections, giving the required imbedding.

Suppose now that M is open. We may suppose that M is connected. Let $(p_i, q_i)(i = 1, 2, \cdots)$ be the points with $f_0(p_i) = f_0(q_i)$. For each i, let C_i be a smooth curve in M from p_i to infinity, and let U_i be a neighborhood of C_i, so that U_i touches no other U_j or q_j. Let

$$(p, r, s)(p \in S_0^{n-2}, 0 \leqq r \leqq 1, 0 \leqq s < 1)$$

represent the points of U_i (supposing $n \geqq 2$), with $r = 0$ and $2\delta \leqq s < 1$ representing C_i; the points with $r = 1$ or $s = 0$ give the boundary of U_i. Deform U_i into itself by setting

$$\phi_t(p, r, s) = [p, r, \{1 - (1 - \delta)(1 - r)t\}s];$$

this leaves the boundary of U_i fixed. It may easily be replaced by a smooth deformation. Doing this for each i gives a mapping f with $f(M) \subset f_0(M) - \sum C_i$. Clearly f has no self-intersections, and is still proper.

PROBLEM. Does there exist an imbedding, for M open, with no limit set?

9. A lemma on bundles of vector spaces

A bundle $\mathfrak{B}$ *of vector spaces* is defined as follows. Let V_0^m be a fixed vector space of dimension n. Let A be a topological space, the *base space*, and let $\{F_i\}$ be a covering of A by closed sets (we could use open sets). To each $p \in A$ there corresponds a set of points $V^m(p)$; $V^m(p) \cap V^m(q) = 0$ if $p \neq q$. For each F_i, and each $p \in F_i$, $\xi_{F_i}(p, q)$ $(q \in V_0^m)$ is a one-one mapping of V_0^m onto $V^m(p)$. Let $q = \xi_{F_i}^{-1}(p, r)$ mean $r = \xi_{F_i}(p, q)$. For $p \in F_i \cap F_j$, we assume $\xi_{F_i}^{-1}(p, \xi_{F_j}(p, q))$ is a linear mapping of V_0^m onto itself, which varies continuously with p. Because of this, each $V^m(p)$ may be considered as a vector space. The ξ_{F_i} are the *coordinate systems*; the set of all points on all $V^m(p)$ is the *total space*. We write $\mathfrak{B} = (\mathfrak{B}; V_0^m, A)$. If there exists a coordinate system $\zeta(p, q)$, defined for p in all of A, then $\mathfrak{B}$ is *simple*; ζ expresses the total space as the topological product of A and V_0^m.

If we replace V_0^m by a sphere S_0^{m-1}, and linear mappings by orthogonal mappings, we have a *sphere bundle* (see [3]).

LEMMA 12. *Let* $(\mathfrak{B}; V_0^m, K)$ *be a bundle with a complex* K *for base space. Let* K' *be a (possibly void) subcomplex of* K. *Let* $e_1, \cdots, e_m$ *be the unit vectors in* V_0^m. *Let* $\zeta(p, e_1), \cdots, \zeta(p, e_{i-1})$ *be defined for* $p \in K$, *and let* $\zeta(p, e_i)$ *be defined for* p *in* K', *so that these are continuous and independent where defined. Then*

$\zeta(p, e_i)$ may be extended continuously over K so as to be independent of the other $\zeta(p, e_i)$ if either of the following two conditions hold:

(a) $\dim (K) \leqq m - i$.

(b) *K can be built up from K' by adding cells, each having a cell, or nothing, in common with the subcomplex of K' plus the cells already chosen.*

REMARKS. In (b), it follows that $\mathfrak{B}$ is simple. If $\dim (K) = m - i + 1$, the existence of $\zeta(p, e_i)$ is expressible simply in terms of one of the "characteristic classes" of $\mathfrak{B}$ (see [3]).

The proof in Case (b) is due essentially to Wazewski.[7] In Case (a), we extend $\zeta(p, e_i)$ over cells of K of higher and higher dimensions. That this can be done at each step is a simple consequence of the dimensions under consideration (see a similar proof in §10), and is fundamental in the existence of the characteristic classes (which exist equally well for bundles of spheres or vector spaces).

10. The 2-cell σ

Choose the C_i as described in §8; this is possible, since $n \geqq 3$. Let M_1 and M_2 be neighborhoods of C_1 and C_2 in M. Let p_{it} $(0 \leqq t \leqq 1)$ move from p_i along C_i to p_i' as t goes from 0 to 1. We can suppose p_{it} is an imbedding of the interval $(0, 1)$ in M. Then $q_{it} = f(p_{it})$ is an imbedding.

Let E^2 have coordinates x, y. Let A_1 be the interval $0 \leqq x \leqq 1, y = 0$. Let A_2 be the arc of a circle of radius 1 and with center at $(1/2, -\sqrt{3}/2)$, which joins the ends r and r' of A_1 and lies in the half plane $y \geqq 0$. Let $A = A_1 \cup A_2$, let τ' be a small neighborhood of A in E^2, and let τ be τ' plus the interior region τ''. We shall find an imbedding ψ of τ in E^{2n} such that

$$\psi(r) = q, \qquad \psi(r') = q', \qquad \psi(A_i) = B_i \qquad (i = 1, 2),$$

and such that $\psi(\tau) \cap f(M) = B$, and no tangent plane to $\psi(\tau)$ at a point q^* of B lies in a tangent plane to $f(M)$ at q^*. We do this first with τ replaced by τ'.

Let T_1^n and T_2^n be the planes tangent to $f(M_1)$ and $f(M_2)$ at q. Let T^2 be the plane tangent to B_1 and B_2 at q; it cuts T_1^n and T_2^n in lines T_1^1 and T_2^1. By a slight deformation of f, we may suppose that f maps a neighborhood of p_i in M_i onto a neighborhood of q in T_i^n, and maps a neighborhood of C_i (extended) in M_i onto a neighborhood of T_i^1 $(i = 1, 2)$. We may map a closed neighborhood $\bar{V}$ of r in τ linearly onto a closed neighborhood of q in T^2 so that a closed neighborhood of r in A_i goes into a closed neighborhood of q in T_i^1 (and hence in B_i). To this end, we suppose the two ends of A_2 have been straightened. Call this mapping ϕ, and define ϕ similarly near r'.

Let r_{it} run from r to r' along A_i, choosing it so that $\phi(r_{it}) = q_{it}$ where ϕ is defined. Let u_{it} be a smooth vector function defined in A_i and with values in τ, such that u_{i0} and u_{i1} point in the forward and backward directions along A_j at r and r' $(j \neq i)$, and letting u_{1t} and u_{2t} turn in the positive and negative senses respectively as t runs from 0 to 1. Then u_{it} is not tangent to A_i at r_{it}. Let

[7] W. Wazewski, Compositio Math. vol. 2 (1935), pp. 63–68.

R_{it} be a segment of length ρ and center r_{it}, in the direction of u_{it}. We may suppose $\bar{V}$ and $\bar{V}'$ chosen so that part of their boundaries are formed by segments R_{it} extended, and may choose ρ so that no R_{1t_1} touches an R_{2t_2} if r_{1t_1} and r_{2t_2} are outside $\bar{V} \cup \bar{V}'$.

Set $v_{it} = \nabla\phi(u_{it}, r_{it})$ at points r_{it} of $\bar{V} \cup \bar{V}'$. Extend v_{it} over the rest of B_i so that it is smooth and not tangent to $f(M_i)$ at q_{it}. This may be done as follows (or we could use Lemma 12, (a)). Extend v_{it} so as to be smooth over all of B_i except a small piece D_i. With a curvilinear coordinate system about D_i, we may suppose D_i lies on the y_1-axis, and $f(M_i)$ lies in the $(y_1, \cdots, y_n)$-plane E^n. Now $v_{it} = v'_{it} + v''_{it}$, v'_{it} and v''_{it} being tangent and orthogonal to E^n respectively; v''_{it} is $\neq 0$ at the ends of D_i. Extend v'_{it} to be smooth over D_i. Since $2n - n > 1 = \dim(D_i)$, we may extend v''_{it} over D_i so as to be smooth and $\neq 0$ there. Now set $v_{it} = v'_{it} + v''_{it}$ in D_i.

Since ϕ is linear so far,

$$\phi(r_{it} + \alpha u_{it}) = q_{it} + \alpha v_{it} \quad (r_{it} \in \bar{V} \cup \bar{V}', |\alpha| \leqq \rho).$$

We may use this formula to extend ϕ over a closed neighborhood $\bar{\tau}'$ of A.

Let ψ' be a continuous extension of ϕ over τ. Since $2n \geqq 5$, we may[8] replace ψ' by an imbedding ψ such that $\psi(r)$ is arbitrarily close to $\psi'(r)$, and as we approach points of A, ψ approaches ψ' closer and closer, together with first derivatives. It follows that ψ is an imbedding, and no tangent plane to $\psi(\tau)$ at a point $\psi(r^*)$ $(r^* \in A)$ lies in a tangent plane to $f(M)$ at $\psi(r^*)$. Further, since $n + 2 < 2n$, we may suppose that $\psi(\tau) \cap f(M) = B$. Now $\sigma = \psi(\tau)$ is the required 2-cell.

11. The neighborhood of σ

It is here we shall use the fact that the two self-intersections are of opposite types. Our present purpose is to define smooth vector functions $w_1(q^*), \cdots, w_{2n}(q^*)$, for $q^* \in \sigma$, so that these are independent, and if $q^* = \psi(r^*)$,

(a) $w_1(q^*) = \nabla\psi(e_1, r^*)$, $w_2(q^*) = \nabla\psi(e_2, r^*)$,

(b) $w_3(q^*), \cdots, w_{n+1}(q^*)$ are tangent to $f(M_1)$ at q^* for $q^* \in B_1$,

(c) $w_{n+2}(q^*), \cdots, w_{2n}(q^*)$ are tangent to $f(M_2)$ at q^* for $q^* \in B_2$.

These show how σ, $f(M_1)$ and $f(M_2)$ lie together in E^{2n}.

Since ψ is regular, $w_1(q^*)$ and $w_2(q^*)$ are independent. Let V_0^{2n} be the space of vectors in E^{2n}. Let $e_3, \cdots, e_{n+1}$ and $e_{n+2}, \cdots, e_{2n}$ determine the subspaces V_1^{n-1} and V_2^{n-1} respectively. Consider, at each point $q^* \in B_1$, the vector space $V_1^{n-1}(q^*)$ of vectors tangent to $f(M_1)$ but orthogonal to B_1. Considering $V_1^{n-1}(q^{**})$ as disjoint from $V_1^{n-1}(q^*)$ if $q^{**} \neq q^*$, we have a bundle of vector spaces $\mathfrak{B}_1$ with the cell B_1 as base space; use $\mathfrak{B}_1 = (\mathfrak{B}_1; V_1^{n-1}, B_1)$. By Lemma 12, (b), we may find a coordinate system ζ_1 in it; choose it so that $w_1(q^*), w_3(q^*), \cdots, w_{n+1}(q^*)$ determine the positive orientation of $f(M)$ at $q^* \in B_1$, if we set

$$w_i(q^*) = \zeta_1(q^*, e_i) \quad (q^* \in B_1; i = 3, \cdots, n+1). \tag{11.1}$$

[8] H. Whitney, Trans. Am. Math. Soc. vol. 36 (1934), pp. 63–89, Lemma 6; compare [1], Theorem 5, or Theorem 2, with the remarks following it.

Since the only vectors at $q^* \in B_1$ which are tangent to both τ and $f(M)$ are tangent to B_1, the w_i are independent so far, and all properties hold.

Similarly, taking $V_2^{n-1}(q^*)$ tangent to $f(M)$ but orthogonal to B_2 at $q^* \in B_2$ gives a bundle $(\mathfrak{B}_2 ; V_2^{n-1}, B_2)$ with a coordinate system ζ_2. Define $w_i(q^*)$ by (11.1), with ζ_2, for $q^* \in B_2$, $i \geqq n + 2$. If $w_2'(q^*)$ points positively along B_2 at q^*, we may suppose $w_2'(q^*)$, $w_{n+2}(q^*)$, $\cdots$, $w_{2n}(q^*)$ determine the positive orientation of $f(M_2)$ at q^*. The properties still hold. (At q and q', clearly all the vectors are independent.)

Let $(\mathfrak{B}; V_0^{2n}, \sigma)$ be the bundle with each $V(q^*) = V_0^{2n}$. (The total space is $V_0^{2n} \times \sigma$.) In the subbundle $(\mathfrak{B}; V_0^{2n}, B_2)$ over B_2, the vectors $w_1(q^*)$, $w_2(q^*)$, $w_{n+2}(q^*)$, $\cdots$, $w_{2n}(q^*)$ are independent; $w_1(q^*)$, $\cdots$, $w_{2n}(q^*)$ are independent for $q^* = q$ and for $q^* = q'$. By Lemma 12, using (a), with $K = B_2$, $K' = q \cup q'$, we may extend $w_3(q^*)$, $\cdots$, $w_n(q^*)$ over B_2 so that the $2n - 1$ vectors are independent. Now recall that the two self-intersections are of opposite types (we are supposing that M is orientable and n is even). By the choice of ζ_1 and ζ_2, this means that

$$w_1(q^*), \quad w_3(q^*), \cdots, w_{n+1}(q^*), \quad w_2'(q^*), \quad w_{n+2}(q^*), \cdots, w_{2n}(q^*),$$

define opposite orientations of E^{2n} for $q^* = q$ and for $q^* = q'$. Now $w_2(q)$ and $w_2(q')$ may be deformed into $w_2'(q)$ and $-w_2'(q')$ respectively, keeping them tangent to σ and independent of B_1; they therefore remain independent of the other vectors listed above. Hence

$$w_1(q^*), w_2(q^*), \cdots, w_{2n}(q^*)$$

define the same orientation of E^{2n} at $q^* = q$ as at $q^* = q'$. It follows that $w_{n+1}(q^*)$ may be defined over B_2 so as to remain independent of the other vectors. (If we deform $w_{n+1}(q^*)$ for q^* near q and q' to become a unit vector orthogonal to the other vectors, then $w_{n+1}(q^*)$ is uniquely determined so as to be continuous and have this property in the rest of B_2.)

By Lemma 12, using (a) again, with $K = \sigma$ and $K' = B$, since $\dim(\sigma) = 2 \leqq 2n - (n + 1)$, we may extend $w_3(q^*), \cdots, w_{n+1}(q^*)$ over σ so that $w_1(q^*), \cdots, w_{n+1}(q^*)$ are independent there. Finally, by Lemma 12, using (b), since σ and B_2 clearly have the property of K and K', we may extend $w_{n+2}(q^*)$, $\cdots$, $w_{2n}(q^*)$ over σ. The $w_i(q^*)$ determine a neighborhood of σ in E^{2n}, as will be apparent from (12.1).

12. Completion of the proof of Theorem 4

Consider $\tau \subset E^2$ as lying in E^{2n}. For each point $(a_1, a_2, a_3, \cdots, a_{2n})$, set $r^* = (a_1, a_2, 0, \cdots, 0)$, and

$$\psi\left(r^* + \sum_{i=3}^{n} a_i e_i\right) = \psi(r^*) + \sum_{i=3}^{n} a_i w_i(\psi(r^*)). \tag{12.1}$$

Since $w_1(q^*), \cdots, w_{2n}(q^*)$ are independent and smooth, this is a mapping with non-vanishing Jacobian; hence, in a neighborhood of the interior of σ, we may

invert, and define ψ^{-1}. (The $w_i(q^*)$ are taken of class C^2 for this to hold.) If we set

$$N_1 = \psi^{-1}(f(M_1)), \qquad N_2 = \psi^{-1}(f(M_2)),$$

then N_1 and N_2 will lie in a neighborhood U of τ. If we define a deformation of N_2 in U which brings it into a position not intersecting N_1, applying ψ defines a corresponding deformation of f; there will then be two self-intersections fewer, as required.

Set $\pi(x_1, \cdots, x_{2n}) = (x_1, 0, x_3, \cdots, x_{2n})$. By (a), (b) and (c) of §11, and (12.1), for any $r^* \in A_1$, the tangent plane to N_1 and hence to $\pi(N_1)$ at r^* lies in the $(x_1, x_3, \cdots, x_{n+1})$-plane, and the tangent plane to $\pi(N_2)$ at r^* lies in the $(x_1, x_{n+2}, \cdots, x_{2n})$-plane, it follows that

(12.2) $\qquad \pi(N_1) \cap \pi(N_2)$ is on the x_1-axis.

Let $\mu(x_1)$ be a smooth real function whose graph $x_2 = \mu(x_1)$ lies just above A_2, and lies on the x_1-axis just outside A_2. Take $\epsilon > 0$ so that the points of N_2 within ϵ of the (x_1, x_2)-plane E^2 form a set interior to N_2. Let $\nu(\lambda)$ be a smooth real function with

$$|\nu(\lambda)| \leqq 1, \qquad \nu(0) = 1, \qquad \nu(\lambda) = 0 \quad \text{for} \quad |\lambda| \geqq \epsilon^2.$$

For $r^* = (x_1, \cdots, x_{2n})$, set

(12.3) $\qquad \theta_t(r^*) = r^* - t\nu(x_3^2 + \cdots + x_{2n}^2)\mu(x_1)e_2,$

and consider this as applying only to points of N_2. This is clearly a regular deformation. For $t = 1$, it moves the part of N_2 lying on A_2 into a position with $x_1 < 0$. Since $\pi(\theta_t(N_2)) = \pi(N_2)$, (12.2) gives

$$N_1 \cap \theta_1(N_2) \text{ is on the } x_1\text{-axis.}$$

But $\theta_1(N_2)$ does not touch the x_1-axis; hence this set is void. Furthermore, since $\psi(\tau) \cap f(M) = B$, taking ϵ small enough will insure that no new self-intersections are introduced. This completes the proof in the case that M is orientable and n is even.

In the other cases, we can get rid of any pair of self-intersections by a regular deformation. To show this, it is only necessary to show that C_1 and C_2 may be chosen in such a fashion that M_1 and M_2, when each is oriented, intersect in points of opposite types.

Suppose M is not orientable. Try a pair C_1, C_2. If the points are of the same type, choose a curve C_2' such that $C_2 \cup C_2'$ reverses the orientation in M; then C_1 and C_2' will do.

Suppose n is odd, and M is orientable. If C_1 and C_2 will not do, choose C_1' and C_2' so that (with the notation of §9) C_1' joins p_1 and p_2', and C_2' joins p_2 and p_1'. We may suppose C_i' coincides with C_i near one end and with C_j $(j \neq i)$ at the other, and $M_i = M_i'$ near p_i, and $M_i = M_j'$ $(j \neq i)$ near p_i'. Orient the M_i, and the M_i' accordingly (near p_i). Then the intersection of the

pair (M_1', M_2') is like that of (M_2, M_1) at q'; hence (compare the proof in Lemma 2) the intersection is of opposite type from that of (M_1, M_2) at q', as required. The theorem is now proved.

Appendix

Imbedding the complex projective plane M^4 in E^7

Let M^6 be the space of all triples (z_1, z_2, z_3) of complex numbers, not all 0. If we identify any two triples (z_1, z_2, z_3), $(\lambda z_1, \lambda z_2, \lambda z_3)$ ($\lambda \neq 0$ complex), M^4 is obtained. Let $[z_1, z_2, z_3]$ denote points of M^4.

We first give briefly a topological description of M^4, then discuss how an imbedding may be found from the topological point of view, and finally give equations which are proved to be an imbedding. In terms of the real variables, the imbedding will be analytic; it of course cannot be so in terms of the complex variables, since 7 is odd.

Set $z_k = x_k + iy_k$, and

$$N(z_1, z_2, z_3) = |z_1|^2 + |z_2|^2 + |z_3|^2 = \sum (x_k^2 + y_k^2). \tag{1}$$

We remark that, since $[z_1, z_2, z_3]$ may be normalized to make $N = 1$, uniquely except for a factor λ with $|\lambda| = 1$, the 5-sphere $N = 1$ is expressed as the total space of a bundle of 1-spheres with M^4 as base space. A similar fact holds for the $(2n + 1)$-sphere.

First we divide M^4 into three subsets; the points with $z_3 \neq 0$, those with $z_3 = 0$ but $z_2 \neq 0$, and those with $z_3 = z_2 = 0$. These sets, with normal forms for their points, are:

$$\left.\begin{aligned} &Q^4\text{: all } [z_1, z_2, 1], \\ &Q^2\text{: all } [z_1, 1, 0], \\ &Q^0\text{: all } [1, 0, 0]; \end{aligned}\right\} S^2.$$

Since these are cells, the Euler-Poincaré characteristic of M^4 is 3. S^2 is complex projective 1-space, equivalent to the extended complex plane, and is topologically a 2-sphere. For any $[z_1, z_2, 0]$ in S^2, set

$$P^*[z_1, z_2, 0] = \text{all } [z_1, z_2, z_3].$$

Set

$$\rho[z_1, z_2, z_3] = |z_3|^2/(|z_1|^2 + |z_2|^2), \tag{2}$$

(ρ may be ∞), and define the subsets of M^4

$$A_-\text{: } \rho < 1, \qquad A\text{: } \rho = 1, \qquad A_+\text{: } \rho > 1.$$

Set $P_-[z_1, z_2, 0] = P^*[z_1, z_2, 0] \cap A_-$, and define P, P_+ similarly.

Clearly A is a 3-sphere (see below), and A_+ is a 4-cell bounded by it; each is in Q^4. Since the sets $P_-(p) \cup P(p)$ are disjoint, these form a bundle of closed 2-cells, with S^2 as base space. The $P(p)$ form a bundle of 1-spheres with S^2

as base space; the total space is the 3-sphere A. We remark that this proves the 2-dimensional invariant of the bundle to have the value ± 1; also, that mapping each point q of A into $p \in S^2$ such that $q \in P(p)$ is exactly Hopf's essential mapping of a 3-sphere into a 2-sphere.[9]

We now discuss the imbedding problem. If we set $F^*[z_1, z_2, 1] = (z_1, z_2)$, this imbeds Q^4 in 4-space. Of course F^* cannot be extended over S^2. To make this possible, let us pull back towards the origin those points with small ρ; we may set

$$F'[z_1, z_2, 1] = \frac{2}{N}(z_1, z_2) \tag{3}$$

or equivalently (the last expression allowable even for $z_3 = 0$)

$$F'[z_1, z_2, z_3] = \frac{2|z_3|^2}{N}\left(\frac{z_1}{z_3}, \frac{z_2}{z_3}\right) = \frac{2}{N}(z_1\bar{z}_3, z_2\bar{z}_3). \tag{4}$$

This mapping fails to be regular at points of A, where the vectors pointing away from [0, 0, 1] (using (3)) are mapped into the origin.

Let us introduce three new coordinates into the image space, map S^2 into the unit sphere in this space, and extend the mapping over M^4. Using stereographic projection, and one complex and one real coordinate, we find

$$F''[z_1, z_2, 0] = \left(2\frac{z_1\bar{z}_2}{|z_1|^2 + |z_2|^2}, \frac{|z_1|^2 - |z_2|^2}{|z_1|^2 + |z_2|^2}\right),$$

or more generally,

$$F''[z_1, z_2, z_3] = \frac{1}{N}(2z_1\bar{z}_2, |z_1|^2 - |z_2|^2).$$

Combining this with (4), with a slight change, we define our mapping by

$$F[z_1, z_2, z_3] = \frac{1}{N}(z_2\bar{z}_3, z_3\bar{z}_1, z_1\bar{z}_2, |z_1|^2 - |z_2|^2), \tag{5}$$

using three complex and one real coordinate, N being defined by (1). In terms of the seven real variables,

$$F = \frac{1}{N}(x_2x_3 + y_2y_3, y_2x_3 - x_2y_3, x_1x_3 + y_1y_3, x_1y_3 - y_1x_3, \tag{6}$$
$$x_1x_2 + y_1y_2, y_1x_2 - x_1y_2, x_1^2 + y_1^2 - x_2^2 - y_2^2).$$

From (5) it is clear that $F[z_1, z_2, z_3] = F[\lambda z_1, \lambda z_2, \lambda z_3]$, λ complex, i.e. it is independent of the manner of writing a point in M^4.

We now prove that F is regular. Consider first the F' of (3). Since the set

[9] H. Hopf, *Über die Abbildungen der dreidimensionalen Sphäre auf die Kugelfläche*, Math. Annalen, vol. 104 (1931), pp. 637–665. The mapping also is that determined in the Remark following (1).

of points (z_1, z_2) with $|z_1|^2 + |z_2|^2 = r^2$, which forms a 3-sphere of radius r if $r > 0$, is mapped into a concentric sphere, F' is regular wherever the vector $v_{z_1 z_2}$ at $[z_1, z_2, 1]$ pointing away from the origin is mapped into a non-zero multiple of itself. Set

$$\alpha = x_1^2 + y_1^2 + x_2^2 + y_2^2, \qquad N_\lambda = 1 + \lambda^2\alpha^2 \quad (\lambda \text{ real}).$$

Then the image of $\frac{1}{2}v_{z_1 z_2}$ is the value when $\lambda = 1$ of

$$\frac{\partial}{\partial\lambda}\left[\frac{1}{N_\lambda}(\lambda x_1, \lambda y_1, \cdots)\right] = -\frac{2\lambda\alpha^2}{N_\lambda^2}(\lambda x_1, \cdots) + \frac{1}{N_\lambda}(x_1, \cdots)$$

$$= -\frac{1}{N_\lambda^2}(2\lambda^2\alpha^2 - N_\lambda)(x_1, y_1, x_2, y_2).$$

Taking $\lambda = 1$, we see that F' is regular except possibly on a set B_3, where $2\alpha^2 - N = 0$, i.e. $2\alpha^2 = 1 + \alpha^2$, $\alpha = 1$, or else $z_3 = 0$. With $[z_1, z_2, z_3]$, we see that B_3 is given by the condition:

(7) Either $z_3 = 0$ or $|z_3|^2 = |z_1|^2 + |z_2|^2$,

i.e. $B_3 = S^2 \cup A$.

Define B_1, B_2 similarly, and set $B = B_1 \cap B_2 \cap B_3$. Then replacing F' by F (where we need not consider the last variable $|z_1|^2 - |z_2|^2$), we see that F is regular except possibly at points of B. Now B contains only the points

(8) $\qquad p_{1z} = [0, 1, z], \qquad p_{2z} = [z, 0, 1], \qquad p_{3z} = [1, z, 0], \qquad |z| = 1.$

For $[0, 0, 1]$ is not in B_3; similarly for $[0, 1, 0]$ and $[1, 0, 0]$. In case at most one $z_i = 0$, say $|z_1| \leqq |z_2| \leqq |z_3|$; then $z_2 \neq 0$. If $z_1 \neq 0$, or if $z_1 = 0$ and $|z_3| > |z_2|$, then $|z_2|^2 < |z_1|^2 + |z_3|^2$, and $[z_1, z_2, z_3]$ is not in B_2, while if $z_1 = 0$ and $|z_3| = |z_2|$, then $[z_1, z_2, z_3] = [0, 1, z_3/z_2] = p_{1z}$, $z = z_3/z_2$.

We must prove still that F is regular at the points p_{iz}. To do this, it is sufficient to show that the $7 \times 6 = 42$ partial derivatives of the components of F, considered as mapping M^6 into E^7, form a matrix of rank 4 at each such point. For this proves that the tangent space to M^6 at any such point is mapped into a space of dimension 4 in E^7 by F; but the tangent space to M^4 is clearly mapped into the same space in E^7. It is of course sufficient to find a submatrix of rank 4.

For points of M^6 representing points p_{iz}, we choose points with $N = 1$, obtained from (8) by multiplying by a real positive factor. Now

$$p_{1z} = (0, 0, x_2, 0, x_3, y_3), \qquad x_2^2 = x_3^2 + y_3^2 = \tfrac{1}{2},$$
$$p_{2z} = (x_1, y_1, 0, 0, x_3, 0), \qquad x_3^2 = x_1^2 + y_1^2 = \tfrac{1}{2},$$
$$p_{3z} = (x_1, 0, x_2, y_2, 0, 0), \qquad x_1^2 = x_2^2 + y_2^2 = \tfrac{1}{2}.$$

The corresponding values of F are:

at p_{1z}: $(x_2x_3, \; -x_2y_3, \; 0, \; 0, \; 0, \; 0, \; -x_2^2)$,

at p_{2z}: $(0, \; 0, \; x_1x_3, \; -y_1x_3, \; 0, \; 0, \; x_1^2 + y_1^2)$,

at p_{3z}: $(0, \; 0, \; 0, \; 0, \; x_1x_2, \; -x_1y_2, \; x_1^2 - x_2^2 - y_2^2)$.

Consider first a point p_{1z}. Form F_1 from F by omitting the third and fourth of the seven variables. Then since $N = 1$ and

$$-2x_2^2x_3 + x_3 = 2x_2^2y_3 - y_3 = 0$$

at the point under consideration, differentiating (6) gives the following matrix for $\partial F_1/\partial x_1, \cdots, \partial F_1/\partial y_2$:

$$\begin{Vmatrix} 0 & 0 & x_2 & 0 & 0 \\ 0 & 0 & 0 & x_2 & 0 \\ 0 & 0 & 0 & 0 & 2x_2(x_2^2 - 1) \\ y_3 & x_3 & 0 & 0 & -2y_2 \end{Vmatrix}.$$

This is obviously of rank 4. Similarly, at p_{2z}, let F_2 omit the fifth and sixth variables from F. Then the matrix of $\partial F_2/\partial x_2, \cdots, \partial F_2/\partial y_3$ is

$$\begin{Vmatrix} x_3 & 0 & 0 & 0 & 0 \\ 0 & x_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2x_3(x_1^2 + x_1^2) \\ 0 & 0 & y_1 & x_1 & 0 \end{Vmatrix},$$

again of rank 4. At p_{3z}, letting F_3 omit the first and second variables, the matrix of $\partial F_3/\partial x_1$, $\partial F_3/\partial y_1$, $\partial F_3/\partial x_3$, $\partial F_3/\partial y_3$ is

$$\begin{Vmatrix} 0 & 0 & 0 & 0 & 2x_1 \\ 0 & 0 & y_2 & x_2 & 0 \\ x_1 & 0 & 0 & 0 & 0 \\ 0 & x_1 & 0 & 0 & 0 \end{Vmatrix},$$

again of rank 4. This completes the proof that F is regular.

To show that F is one-one, take any point

$$q = (w_1, w_2, w_3, \gamma) = (u_1, v_1, \cdots, v_3, \gamma) = F(p)$$

for some $p \in M^4$; we shall show that $p = [z_1, z_2, z_3]$ is uniquely determined, i.e. the z_i are determined up to a complex factor.

Suppose first that $w_3 = 0$, $\gamma = 0$. Then one of z_1, z_2 is 0, and since $|z_1|^2 - |z_2|^2 = 0$, both are 0; hence p is unique.

Suppose next that $w_3 = 0$, $\gamma \neq 0$. If $\gamma > 0$, then $z_2 = 0$, $|z_1| > 0$ (since $N > 0$). Since $z_1 \neq 0$, we can normalize p to $[1, 0, z_3]$. Now

$$\gamma = \frac{|z_1|^2}{N} = \frac{1}{N}, \qquad w_2 = \frac{z_3\bar{z}_1}{N} = \gamma z_3;$$

hence $z_3 = w_2/\gamma$ is determined. If $\gamma < 0$, then $z_1 = 0$, $z_2 \neq 0$, etc.

Suppose now that $w_3 \neq 0$; then $z_1 \neq 0$, $z_2 \neq 0$. In this case we normalize so that

$$N = 1, \qquad z_2 = x_2 > 0.$$

If first $w_2 \neq 0$, then

$$w_1w_3 = x_2^2z_1\bar{z}_3 = x_2^2\bar{w}_2, \qquad x_2^2 = \frac{w_1w_3}{\bar{w}_2},$$

determining x_2. Now

$$z_3 = \frac{\overline{w}_1}{x_2}, \qquad z_1 = \frac{w_3}{x_2},$$

so that p is determined.

Suppose finally that $w_3 \neq 0$, $w_2 = 0$. Normalize as before. Now $z_3 = 0$; we find, from $w_3 = x_2 z_1$, $\gamma = |z_1|^2 - x_2^2$,

$$|z_1|^2 = \frac{|w_3|^2}{x_2^2} = \gamma + x_2^2,$$

$$x_2^2 = \tfrac{1}{2}[-\gamma \pm (\gamma^2 + 4|w_3|^2)^{1/2}].$$

Since the square root is $> |\gamma|$, the minus sign is impossible, and x_2^2 and hence $z_2 = x_2$ is uniquely determined, as is $z_1 = w_3/x_2$. This completes the proof.

HARVARD UNIVERSITY.

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THE SINGULARITIES OF A SMOOTH n-MANIFOLD IN $(2n-1)$-SPACE*

BY HASSLER WHITNEY

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1. Introduction

We showed in the preceding paper that any smooth n-manifold M^n may be imbedded in $2n$-space E^{2n}. Our primary purpose here is to show that it may be immersed in E^{2n-1}, provided that $n \geqq 2$. Then near any point of M, the mapping f into E^{2n-1} is one-one, but there may be self-intersections (which may be required to lie along curves). Equally important perhaps is the combinatorial study of singularities (points where the mapping is not regular). Along with true manifolds we study also manifolds with boundary. By a *partial manifold*, we mean a manifold with or without boundary. In simple cases, the boundary ∂M of the partial manifold M will be a manifold. (∂M means the point set boundary; it need not coincide with the boundary of the chain M if M is non-orientable.) Since the question of how general ∂M may be allowed to be (we insist at any rate that it be a complex) is a rather difficult one, which we expect to study further in another paper, we will use the term somewhat loosely here. Any special assumptions on ∂M which may be needed will be made at the time.

It is a highly difficult problem to see if the imbedding and immersion theorems of the preceding paper and the present one can be improved upon. Practically the only knowledge we have of this is found in the author's Michigan lecture, [3]. The most important result there for the present problem is the existence of a closed M^4 which cannot be imbedded in E^7. (We have not studied the possibility of immersing it in E^6.) This M^4 is non-orientable; it seems possible that any orientable, or any open or partial, M^4 may be imbedded in E^7, and immersed in E^6. Possibly also any M^3 may be imbedded in E^5!

We touch briefly on the case $n = 1$. Here M is a circle, or a closed, open, or half-open arc. Locally, the mapping f into the line E^1 is expressible as a differentiable real-valued function $x' = f(x)$; the singularities of f are the points where $df/dx = 0$. If the mapping is "semi-regular," the only singularities are maxima and minima of f. It is obvious that a slight alteration of any f will give a mapping g in which this holds. The combinatorial conditions as stated in this paper apply only to the case $n \geqq 2$, but analogues could easily be given for the case $n = 1$.

Suppose now that $n \geqq 2$. If we take a general smooth mapping f of M^2 into E^3, the singularities may be quite wild. But again, a slight alteration of f will reduce them to a single type; see §3 and Fig. 1. The new mapping is semi-regular; these are the mappings which concern us here.

The combinatorial part of the paper consists essentially in counting the

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algebraic number $\mathfrak{L}_f(M)$ of singular points (mod 2 if n is even) by means of the mapping f in the boundary of M. (As an immediate corollary, $\mathfrak{L}_f(M) = 0$ or $\equiv 0 \bmod 2$ for all closed manifolds, orientable or non-orientable.) The manner of counting may be seen from Fig. 1. If we follow around the boundary ∂M, it cuts through M at one point, p^*; hence $\mathfrak{L}_f(M) \equiv 1 \pmod 2$. Or, again, let us cut off a strip around the edge of $f(M)$, and spread it out in the form of a circle; we will find that it has a single twist. If we cut this strip into two strips, the twist will show up in the linking of the two new strips; combinatorially, in the fact that (each being considered as a curve) their looping coefficient is ± 1. These two facts are expressible in the form:

$$\mathfrak{L}_f(M) = KI(fM^*, f\partial M) = LC(f\partial M^*, f\partial M), \tag{1.1}$$

M^* being M with a narrow neighborhood of ∂M removed. (1.1) holds in fact for any chain A, using A^*, which is A with a neighborhood of all $(n-1)$-cells removed. The fundamental theorem states that this is the algebraic number of singular points of M, taken mod 2 if n is even. The proof in the case that M is non-orientable and n is odd is difficult to handle; the intersection theory in Part III is needed to unravel the situation.

The reader may wonder why we are willing to lose preciseness in the results by reducing mod 2 whenever n is even. The answer is, the formulas are only correct after reducing mod 2 (and in fact, this is all we need in the proof of the immersion theorem). This is well illustrated in Fig. 2. There are no singularities; yet $f(\partial M)$ cuts through $f(M)$ twice, each time in the same sense, so that $KI(fM^*, f\partial M) = \pm 2$. In the proof of the immersion theorem, we cut out pieces of $f(M)$ and alter them; it may be necessary to insert twisted pieces such as in Fig. 2 to gain the desired end in case n is even.

Fundamental definitions are as in the preceding paper (including its §4). We note also the following (see also §15 and elsewhere). A vector is *tangent* to M at $p \in M$ if it points into M or along ∂M (if $p \in \partial M$) at p. It is *independent* of M if it is not in the tangent plane to M at p. A vector field $f(p)$ is *independent* of M if each $v(p)$ is independent of M at p. We may use a complex K in place of M; then v is *independent* of K at p if it is independent of each cell σ of K with $p \in \bar{\sigma}$, etc.

2. Outline of the paper

A typical singularity is presented in (3.3), and a mapping of a sphere or plane with just two of them is given in §4. Though there is only one kind of singular point under a semi-regular mapping, we may differentiate between positive and negative ones in case n is odd; see the definition in §5. If a single cell is mapped by f so that it has just one singularity, as illustrated in Fig. 1, the relation to intersections is fairly simple, as noted above. For a partial manifold, or more generally, a complex, the relation is worked out in §7. When we express M and ∂M as chains, and sum, both pairs $\partial\sigma_i^n$, σ_j^n and $\partial\sigma_j^n$, σ_i^n will appear if $i \neq j$. By the commutation rule for Kronecker indices, such terms will cancel out for

n odd, giving the exact value of $\mathfrak{L}_f(M)$ in (8.2); for n even, we get this result only mod 2. If f is deformed, $f(\partial M)$ may cut through itself; yet if n is odd, this does not affect $\mathfrak{L}_f(M)$, as noted in Theorem 4.

To prove the immersion theorem, we need some detailed results on the type of looping coefficients which we mentioned above in cutting a strip into two strips. Lemmas are given which state that certain alterations of f are possible which map ∂M into a given position and let M have given directions at points of ∂M. Next we give a mapping f of an n-cube M_0 (n even) without singularities and with $\mathfrak{L}_f(M_0) = \pm 2$; compare Fig. 2. To prove the immersion theorem in case M is closed and n is odd, we use $\mathfrak{L}_f(M) = 0$ to show that the singular points may be paired, p_i and p_i', the two in a pair being of opposite type. If A is an arc from p_i to p_i', a neighborhood M_1 of A then has the property that $\mathfrak{L}_f(M_1) = 0$. We may therefore alter f in M_1 to remove these two singularities. The other cases do not require much further treatment. Two theorems are then given which discuss the position of ∂M under an immersion of a partial manifold.

Suppose M is non-orientable. Let M_1 be a chain formed by adding together the n-cells of M. Then $\partial M_1 = A + 2B$, where A is the sum of the $(n-1)$-cells of ∂M, and B is a sum of cells interior to M. Thus, if M^2 is a Möbius strip, ∂M_1^2 is the boundary curve plus twice an arc cutting across the strip. Suppose n is odd. It was proved in Part I that (1.1) counts the singularities, provided that M_1, M_1^* and ∂M_1 are used. But we do not wish to use any interior cells of M; it is necessary to show that these always cancel out. It is clear that in this case we cannot use $KI(fM^*, f\,\partial\, M)$, since by an alteration of f we might move $f\,\partial\, M$ across itself, which would alter the Kronecker index. Moreover, $LC(f\,\partial\, M^*, f\,\partial\, M)$ is not defined, since ∂M^* and ∂M cannot be made into cycles. But if we choose that n-chain $\rho^+ f\,\partial\, M$ formed by deforming $f\,\partial\, M$ in the y_{2n-1}-direction to infinity, we may study $KI(f\,\partial\, M^*, \rho^+ f\,\partial\, M)$. This leads finally to the required result. The definition of $\mathfrak{L}_f(M)$ required, in (20.5), is more complicated than before; its necessity is shown by an example in §21.

In an appendix we take up some topics which are less fundamental in the paper.

I. SINGULARITIES AND INTERSECTIONS

3. The general type of singularity

DEFINITION. The mapping f of the n-manifold M^n (without boundary) into E^{2n-1} is *semi-regular* if it is of class C^{12} (so that we may apply Lemma 2) and is proper, and for each $p \in M$, either f is regular at p or the following holds: With a suitable coordinate system about p,

$$\frac{\partial f}{\partial x_1}\Big|_p = 0, \tag{3.1}$$

and the $2n-1$ vectors

$$\frac{\partial^2 f}{\partial x_1^2}\Big|_p, \frac{\partial f}{\partial x_2}\Big|_p, \cdots, \frac{\partial f}{\partial x_n}\Big|_p, \frac{\partial^2 f}{\partial x_1 \partial x_2}\Big|_p, \cdots, \frac{\partial^2 f}{\partial x_1 \partial x_n}\Big|_p \tag{3.2}$$

are independent. This condition holds then in any coordinate system for which (3.1) holds; see [4]. The x_1-direction is uniquely determined except as to sense. If M is a partial manifold, we assume also that f is one-one in a neighborhood of the boundary. (See the appendix, Lemma 25.)

DEFINITION. The semi-regular mapping f is *completely semi-regular* if: (a) For any double point $f(p) = f(q)$ (p or q may be in ∂M) the two tangent planes to $f(M)$ there have only a line in common. (b) $f(\partial M)$ does not contain the image of any singular point. (c) If $n \geqq 3$, there are no triple points $f(p) = f(q) = f(r)$; if $n = 2$, there is no such triple point with $p \in \partial M$, and there are no quadruple points. The self-intersections are then along smooth curves; see §22.

LEMMA 1. *Arbitrarily close to any f there is a completely semi-regular f'; we may make f' be one-one in a neighborhood of ∂M, and may make f' of class C^∞.*

This is proved without the "completely," for manifolds, in [4]. For partial manifolds, we first imbed a neighborhood of the boundary (using the methods in [1]), then extend the mapping over the interior of M, and apply the proof mentioned to the interior. It is now easy to make the mapping completely semi-regular (see [1], especially §9, (D)).

LEMMA 2. *Let f be semi-regular. Then for any singular point p there exist (curvilinear) coordinate systems $(x_1, \cdots, x_n)$ about p and $(y_1, \cdots, y_{2n-1})$ about $f(p)$ such that f is given near p by*

$$\left.\begin{aligned} y_i &= x_i, \\ y_1 &= x_1^2, \\ y_{n+i-1} &= x_1 x_i, \end{aligned}\right\} \quad (i = 2, \cdots, n). \tag{3.3}$$

This also is proved in [4]. If f is of class C^{4r+8}, of class C^∞, or analytic, the new coordinate systems will be of class C^r, of class C^∞, or analytic respectively.

In case $n = 2$, the mapping is

$$x = u^2, \qquad y = v, \qquad z = uv; \tag{3.4}$$

eliminating u and v gives $z = \pm y\sqrt{x}$. For each y, the cross-section is a parabola; as y passes through 0, the parabola degenerates to a half-ray, and opens out again (with sense reversed); see Figure 1. The only self-intersection is at $v = 0$, mapping into the positive x-axis; in the general case, at $x_2 = \cdots = x_n = 0$, mapping into the positive y_1-axis.

4. A mapping of a sphere or plane with just two singularities

The examples we give here not only are interesting as illustrating mappings of whole manifolds with definite singularities, but are useful in the proof of the fundamental Theorem 6, in the case of an open manifold.

Let S_0^n be the n-sphere $x_1^2 + \cdots + x_{n+1}^2 = 1$ in E^{n+1}. We define a smooth mapping f of E^{n+1}, and hence of S_0^n, into E^{2n-1} by the equations

$$\left.\begin{aligned} y_i &= x_i, \\ y_1 &= x_{n+1}, \\ y_{n+i-1} &= x_1 x_i, \end{aligned}\right\} \quad (i = 2, \cdots, n). \tag{4.1}$$

For $n = 2$, transposing terms gives

$$f(x, y, z) = (x, xy, z).$$

The effect of f is to turn the part $x < 0$ of the sphere inside out. More explicitly, f squeezes the cross-sections S_x^1 for each x so that for $x = 0$, the circle S_x^1 turns into a line segment, and for $x < 0$, into an ellipse with sense reversed. There are obviously two singularities, at $(0, 0, \pm 1)$.

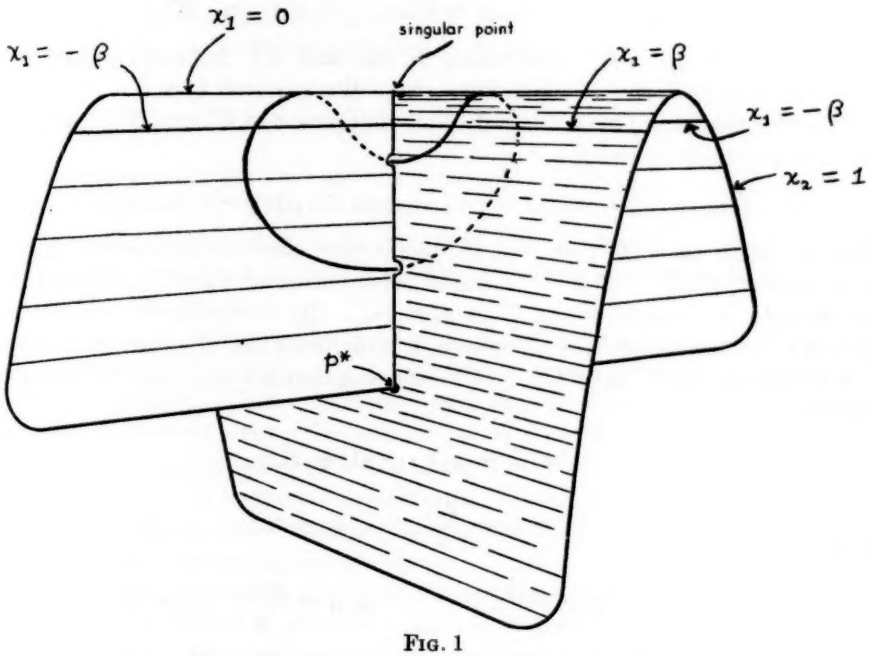

FIG. 1

The matrix of first partial derivatives in (4.1), transposed, is

$$\begin{Vmatrix} 0 & 0 & \cdots & 0 & x_2 & \cdots & x_n \\ 0 & 1 & \cdots & 0 & x_1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & x_1 \\ 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{Vmatrix}.$$

To find the singularities, take any $p = (x_1, \cdots, x_{n+1}) \in S_0^n$, and any vector $v = (v_1, \cdots, v_{n+1})$ tangent to S_0^n at p; then v is orthogonal to $p - O$ (O = origin), so that $\sum v_i x_i = 0$. The vector v is mapped by f into a vector we shall call $\nabla f(v, p)$ (which may be considered as the derivative of f along v; see the preceding paper §4); it is

$$\nabla f(v, p) = \sum v_i \frac{\partial f}{\partial x_i}$$

$$= (v_{n+1}, v_2, \cdots, v_n, v_1 x_2 + v_2 x_1, \cdots, v_1 x_n + v_n x_1).$$

Suppose this vanishes, with $v \neq 0$. Then $v_2 = \cdots = v_n = v_{n+1} = 0$, hence $v_1 \neq 0$, and since $v_1x_i + v_ix_1 = v_1x_i = 0$ $(i > 1)$, we have $x_2 = \cdots = x_n = 0$. Also $\sum v_ix_i = v_1x_1 = 0$, and hence $x_1 = 0$, and $x_{n+1} = \pm 1$. Thus the only singular points are

$$p_1 = (0, \cdots, 0, 1), \qquad p_2 = (0, \cdots, 0, -1).$$

Near each p_k we may determine x_{n+1} in terms of $x_1, \cdots, x_n$, and thus write

$$f(x_1, \cdots, x_n, x_{n+1}) = F(x_1, \cdots, x_n) \quad \text{in} \quad S_0^n;$$

we find $\partial F/\partial x_1 \,|_{p_k} = 0$. Computing $\partial F/\partial x_i$ and $\partial^2 F/\partial x_1 \partial x_i$ at p_1 and at p_2 shows at once that these singularities are of the required type.

From the mapping f in (4.1) we obtain a mapping ϕ of E^n into E^{2n-1} as follows. First, interchange x_1 and x_{n+1}:

$$f_1(x_1, \cdots, x_n, x_{n+1}) = (x_1, \cdots, x_n, x_{n+1}x_2, \cdots, x_{n+1}x_n).$$

Near the point $p_0 = (0, 1, 0, \cdots, 0)$, this is very close to the identity mapping of S_0^n into $S_0^n \subset E^{n+1} \subset E^{2n-1}$; a slight deformation of f_1 into f_2 will bring it to the identity in a neighborhood U of p_0 in S_0^n. By stretching $U - p_0$ into the part of E^n outside some $(n-1)$-sphere, f_2 transforms into the required mapping.

A mapping f of E^n into E^{2n-1} with two singularities may also be defined as follows:

$$u = (1 + x_1^2) \cdots (1 + x_n^2),$$

$$y_1 = x_1 - \frac{2x_1}{u}, \qquad y_i = x_i \qquad (i = 2, \cdots, n), \tag{4.2}$$

$$y_{n+1} = \frac{1}{u}, \qquad y_{n+i} = \frac{x_1 x_i}{u} \qquad (i = 2, \cdots, n-1).$$

Note that far from the origin, f is very near the identity. Hence a slight alteration of f will make it the identity outside some sphere. (Compare the proof of Lemma 11.) Comparing with the preceding paper, §2, we see easily that f is regular except at the points $p_\pm = (0, \cdots, 0, \pm 1)$; at these points, $\partial f/\partial x_1 = 0$. At p_+ for example, the vectors (3.2) form a diagonal determinant, whose elements d_i are $d_1 = 2$, $d_i = 1$ $(i = 2, \cdots, n)$, $d_{n+1} = -1$, $d_{n+i} = \frac{1}{2}$ $(i = 2, \cdots, n-1)$; hence f is semi-regular.

5. The orientation of singular points

We shall discuss the following problem. Given a singular point p and a neighborhood U of p, are there any orientation properties of E or of U determined by the set of points $f(U)$? Let A be the arc of self-intersection through p (i.e. part of the x_1-axis, in the coordinate system of Lemma 2). It turns out that *for n odd, an orientation of E is determined, while for n even, an orientation of E is determined by one of A near p.* We shall show in fact that the following definitions are permissible.

DEFINITIONS. We use the above notations. If n is odd, the singular point is *positive* or *negative* according as the vectors (3.2) determine the negative or positive orientation of E. If n is even, the *positive side* of M at p is the direction along A such that, if the x_1-axis points in that direction, then the vectors (3.2) determine the negative orientation of E. Note that *M need not be oriented or even orientable.* The reason for the choice will appear in Lemma 6.

LEMMA 3. *The above definitions are independent of the coordinate systems employed.*

Take two systems $\{x_i\}$ and $\{x_i'\}$, with $\partial f/\partial x_1 \,|_p = \partial f/\partial x_1' \,|_p = 0$. We may rotate the x_i'-axes ($i > 1$), obtaining $\{x_i''\}$, so that

$$\left.\frac{\partial f}{\partial x_i''}\right|_p = \alpha_i \left.\frac{\partial f}{\partial x_i}\right|_p, \qquad \alpha_i > 0 \text{ for } i > 2.$$

The definitions with the $\{x_i''\}$ are the same as with the $\{x_i'\}$.

If $n \geqq 2$ and $\alpha_2 < 0$, let us replace x_2'' by $-x_2''$. This does not affect the orientation of A, and since both $\partial f/\partial x_2'' \,|_p$ and $\partial^2 f/\partial x_1'' \partial x_2'' \,|_p$ are reversed in direction, the vectors (3.2) with the new $\{x_i''\}$ determine the same orientation of E as with the old $\{x_i''\}$.

Now if $\alpha_1 < 0$, replace x_1'' by $-x_1''$. Suppose first that n is odd. Then the $n - 1$ vectors $\partial^2 f/\partial x_1'' \partial x_2'' \,|_p$, $\cdots$, $\partial^2 f/\partial x_1'' \partial x_n'' \,|_p$ are reversed, but no others of (3.2) are changed. Thus the same orientation of E is determined. Suppose next that n is even. Then the orientation of E is reversed; but the new x_i''-axis now points in the other direction along A.

With all the $\alpha_i > 0$, we may deform the $\{x_i''\}$ system into the $\{x_i\}$ system. The vectors (3.2) remain always independent, so the same orientation of E and of A are determined, completing the proof.

LEMMA 4. *If f is given by* (3.3), *and the coordinate systems determine the positive orientations of M and of E, then for n odd, the singular point is negative, while for n even, the positive direction in M at p is along the negative x_1-axis.*

It is sufficient to show that the matrix formed from the vectors (3.2) has a positive determinant. Differentiating (3.3), we see that the determinant is diagonal, with one 2 and the rest 1's on the diagonal; hence the determinant is $2 > 0$.

6. The intersection $\mathfrak{L}_f(M)$ of M with ∂M under f

We shall count the number of times that $f(\partial M)$ cuts through $f(M)$ in E^{2n-1}. The definition given here suffices in the orientable case; an interpretation in terms of the manner in which M attaches to ∂M will be studied in §9. The latter discussion will apply also to non-orientable partial manifolds such that ∂M can be made into a cycle; see §14.

If A^r and B^s are singular chains in an oriented E^{r+s}, such that $A \cap \partial B = \partial A \cap B = 0$, then their Kronecker index $KI(A, B)$ is defined. In particular, let σ^r and σ^s be oriented cells with just one common interior point p, their tangent planes at p having only p in common. Let $u_1, \cdots, u_r$ be independent vectors

tangent to σ^r at p, determining the positive orientation of σ^r; choose $v_1, \cdots, v_s$ similarly for σ^s. Then the intersection is positive or negative ($KI = 1$ or -1) according as $u_1, \cdots, u_r, v_1, \cdots, v_s$ determine the positive or negative orientation of E^{r+s}.

DEFINITION. Let f be a semi-regular mapping of the orientable partial manifold M^n into E^{2n-1}. Choose an orientation of M. With an infinite subdivision of $M - \partial M$, we obtain an infinite singular chain $M^\odot$. The boundary of M is oriented, and becomes a chain ∂M. We define

$$\mathfrak{L}_f(M) = KI(f\partial M, fM^\odot) = KI(fM^\odot, f\partial M), \tag{6.1}$$

if this is finite. If ∂M is compact, it will be finite, since f is proper.

We now give the definition without the help of the infinite chain $M^\odot$.

LEMMA 5. *Let U be a neighborhood of ∂M in which f is one-one. Let M^* be a singular chain such that*

$$M - M^* \subset U. \tag{6.2}$$

Then (LC = looping coefficient)

$$\mathfrak{L}_f(M) = KI(fM^*, f\,\partial\, M) = LC(f\,\partial\, M^*, f\,\partial\, M). \tag{6.3}$$

For we can write $M^\odot = M^* + M'$ where $M' \subset U$; since f is one-one in U, $KI(fM', f\,\partial\, M) = 0$.

LEMMA 6. *Let the situation be as in Lemma 2. Let σ be an oriented n-cell, lying in the coordinate system, and obtained from the sphere $\sum x_i^2 = \beta^2$ plus interior by cutting off the part with $x_1 > \alpha$, where $0 < \alpha < \beta$. Then for n odd, $\mathfrak{L}(\sigma) = 1$ or -1 according as the singular point is positive or negative, while for n even, $\mathfrak{L}(\sigma) = 1$ or -1 according as the x_1-axis extends in the positive or negative direction in M.*

REMARKS. It is easily seen that for n odd, $\mathfrak{L}(\sigma_1)$ is the same as $\mathfrak{L}(\sigma)$ if σ_1 is obtained from the sphere by cutting off the part $x_1 < -\alpha$; this follows also directly from Theorem 4. In the proof of Theorem 6 for n even, we need only the obvious fact that $\mathfrak{L}(\sigma) = \pm 1$.

Let $p = (-\alpha, 0, \cdots, 0)$, $q = (\alpha, 0, \cdots, 0)$. The only intersection of $f(\partial M)$ with $f(M^\odot)$ is $f(p) = f(q)$. It is clear from Lemma 3 that we may suppose that the coordinate systems determine the positive orientations of M (or σ) and of E. Now the result of Lemma 4 holds, so that it is sufficient to show that $\mathfrak{L}(\sigma) = -1$.

Let $e_1, \cdots, e_n$ be the unit vectors in E^n; these determine the positive orientation of σ, while at q, the vectors $e_2, \cdots, e_n$ determine the positive orientation of $\partial\sigma$. We must show that the vectors

$$\frac{\partial f}{\partial x_1}\bigg|_p, \cdots, \frac{\partial f}{\partial x_n}\bigg|_p, \frac{\partial f}{\partial x_2}\bigg|_q, \cdots, \frac{\partial f}{\partial x_n}\bigg|_q \tag{6.4}$$

determine the negative orientation of E^{2n-1}. The two sets of vectors give the matrices

$$\begin{Vmatrix} -2\alpha & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & -\alpha & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & -\alpha \end{Vmatrix}, \qquad \begin{Vmatrix} 0 & 1 & \cdots & 0 & \alpha & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & \alpha \end{Vmatrix}.$$

Putting the second below the first forms a determinant D which we must prove negative. Subtracting the i^{th} row from the $(n + i - 1)^{\text{th}}$ row $(i = 2, \cdots, n)$ gives a determinant with zeros below the diagonal, whose value is

$$D = (-2\alpha)(2\alpha)^{n-1} = -(2\alpha)^n < 0,$$

as required.

7. The self-intersection of an n-complex mapped into E^{2n-1}

We shall consider mappings of a finite n-complex K into E^{2n-1} *which are one-one in* U for some neighborhood U of K^{n-1}. Of course K might be a subcomplex of a complex of higher dimension. We note that any mapping may be approximated to by one of the required type, even if the cells of K are replaced by more general bounded smooth manifolds; see §16. The considerations will be considerably generalized in Part III.

Let K' be a subdivision of K such that any cell of K' with a vertex in K^{n-1} lies in U. For each oriented σ_i^n of K, let σ_i^* be the sum of the similarly oriented n-cells of K' in σ_i^n which do not touch $\partial\sigma_i^n$. For any chain $A^n = \sum a_i\sigma_i^n$ set $A^* = \sum a_i\sigma_i^*$. The coefficients a_i are integers.

DEFINITION. Generalizing the definition in §6, we set

$$\mathfrak{L}_f(A^n) = KI(fA^*, f\partial A^n) = LC(f\partial A^*, f\partial A^n). \tag{7.1}$$

Note that, if $M = \sum \sigma_i^n$, then $M^* = \sum \sigma_i^*$, which is not the M^* previously used; but the two definitions of $\mathfrak{L}_f(M)$ agree, as is apparent from (8.1) below.

LEMMA 7. *Under the above conditions, we have the point set relations*

$$f(\partial\sigma_i^n) \cap f(\partial\sigma_i^*) = 0, \tag{7.2}$$

$$f(\sigma_i^n - \sigma_i^*) \cap f(\partial\sigma_j^*) = 0 \quad \textit{for} \quad i \neq j. \tag{7.3}$$

For $\partial\sigma_i^n$ and $\partial\sigma_i^*$, also $\sigma_i^n - \sigma_i^*$ and $\partial\sigma_j^*$, are disjoint point sets in U.

THEOREM 1. *Let the mapping f of K into E^{2n-1} be one-one in $K^n \cap U$, U a neighborhood of K^{n-1} in K. Then for any n-chain $\sum a_i\sigma_i^n$,*

$$\mathfrak{L}_f(\sum a_i\sigma_i^n) = \sum a_i^2\mathfrak{L}_f(\sigma_i^n) \qquad \textit{for } n \textit{ odd}, \tag{7.4}$$

$$\mathfrak{L}_f(\sum a_i\sigma_i^n) \equiv \sum a_i\mathfrak{L}_f(\sigma_i^n) \pmod 2 \qquad \textit{for } n \textit{ even}. \tag{7.5}$$

REMARK. We could allow double points $f(p) = f(q)$ with both p and q in

K^{n-1}, for this would not destroy the relations (7.2) and (7.3). But $\mathfrak{L}_f$ would not then be invariant under slight deformations.

We may suppose $K = K^n$ in the proof. Set $\tau_i^n = f(\sigma_i^n)$, $\tau_i^* = f(\sigma_i^*)$. First, by (7.2) and (7.3), $KI(\tau_i^* , \partial\tau_j^n)$ has meaning for all i and j. Hence

$$\mathfrak{L}_f(\sum_i a_i\sigma_i^n) = KI(\sum_i a_i\tau_i^* , \partial \sum_j a_j\tau_j^n) = \sum_{i,j} a_i a_j KI(\tau_i^* , \partial\tau_j^n)$$

$$= \sum_i a_i^2 KI(\tau_i^* , \partial\tau_i^n) + \sum_{i<j} a_i a_j[KI(\tau_i^* , \partial\tau_j^n) + KI(\tau_j^* , \partial\tau_i^n)].$$

Now by (7.3), $\partial\tau_j^*$ does not intersect $\tau_i^n - \tau_i^*$ if $i \neq j$, and $\partial\tau_j^n - \partial\tau_j^*$ bounds a chain $\tau_j^n - \tau_j^*$ which does not intersect the boundary of τ_i^*. Hence, by elementary properties of the Kronecker index,

$$KI(\tau_j^* , \partial\tau_i^n) = (-1)^n KI(\partial\tau_j^* , \tau_i^n) = (-1)^n KI(\tau_i^n , \partial\tau_j^*)$$

$$= (-1)^n KI(\tau_i^* , \partial\tau_j^*) = (-1)^n KI(\tau_i^* , \partial\tau_j^n).$$

Consequently

$$\mathfrak{L}_f(\sum_i a_i\sigma_i^n) = \sum_i a_i^2 \mathfrak{L}_f(\sigma_i^n) + [1 + (-1)^n] \sum_{i<j} a_i a_j KI(\tau_i^* , \partial\tau_j^n), \tag{7.6}$$

from which the theorem follows.

8. Relation between singularities and self-intersections in M

We are now ready to prove the fundamental combinatorial theorem in the orientable case. It will be the primary object of Part III to prove the theorem in the non-orientable case; if n is odd, we need a new definition, (20.5), of $\mathfrak{L}_f(M)$, and a slight further restriction on f.

THEOREM 2. *Let f be a semi-regular mapping of the compact partial manifold M^n into E^{n-1}. Then for n odd, $\mathfrak{L}_f(M)$ is the algebraic number of singular points, while for n even, it is congruent to this number* mod 2.

REMARK. We could find an exact expression for $\mathfrak{L}_f(M)$ when n is even with the help of the classification of self-intersections in §22.

First replace f by a completely semi-regular mapping (Lemma 1), which we again call f. Let $p_1 , \cdots , p_\mu$ be the singular points. About each p_i choose a cell σ_i^n as in Lemma 6. We may let these be cells of a subdivision of M (which need not be simplicial) into smooth cells.[1] Moreover, by first deforming the cells of the subdivision slightly so that $(n-1)$-cells touch the curves of self-intersection in interior isolated points only, and then deforming slightly again, we may obtain a subdivision K such that f is one-one over K^{n-1}. By Lemma 25, it is one-one in a neighborhood of K^{n-1}. Also, since $M - \sum_{i=1}^{\mu} \sigma_i^n$ contains no singular points, we may suppose that f is one-one in each σ_i^n, $i > \mu$.

Define K' and the σ_i^* as in §7, and M^* as in §6. Now

$$f(\partial M) \cap f[\sum(\sigma_i^n - \sigma_i^*) \cap M^*] = 0,$$

[1] Rather than prove this fact, we could easily construct a subdivision containing cells $\sigma_i'^n$ approximately like the cells σ_i^n, and note that $\mathfrak{L}(\sigma_i'^n) = \mathfrak{L}(\sigma_i^n)$.

and hence

$$KI(fM^*, f\partial M) = KI(\sum \sigma_i^*, f\partial M). \tag{8.1}$$

Applying theorem 1 gives

$$\begin{aligned} \mathfrak{L}_f(M) &= \sum_{i=1}^{\mu} \mathfrak{L}_f(\sigma_i^n) && \text{for } n \text{ odd,} \\ \mathfrak{L}_f(M) &\equiv \sum_{i=1}^{\mu} \mathfrak{L}_f(\sigma_i^n) \pmod 2 && \text{for } n \text{ even,} \end{aligned} \tag{8.2}$$

since $\mathfrak{L}(\sigma_i^n) = 0$ for $i > \mu$, as f is one-one there. The theorem now follows from Lemma 6.

THEOREM 3. *Let f be a semi-regular mapping of the closed manifold M^n into E^{2n-1}. Then for n odd, the algebraic number of singular points vanishes, while for n even, it vanishes* mod 2.

This is a corollary of the last theorem.

EXAMPLE. For $n = 1$, the theorem says that a (semi-regular) real-valued function defined on a circle has the same number of maxima as minima.

We give finally an invariance theorem whose proof requires Lemma 9 below.

THEOREM 4. *Let f_t be a deformation of the compact partial manifold M such that f_0 and f_1 are semi-regular, and for some neighborhood U of ∂M, each f_t is regular in U. Then if n is odd, f_0 and f_1 have the same algebraic number of singular points, while for n even, they have the same number* mod 2.

The hypothesis on f_t shows that each $\mathfrak{L}_{f_t}(M)$ may be defined as in §9. By Lemma 9, it is constant for n odd, and is constant (mod 2) for n even. The theorem now follows from Theorem 2.

9. Looping coefficients of vector fields in manifolds in space

DEFINITIONS. Let K^r be a finite smooth complex in E^{2r+1}, and let $v(p)$ be a continuous vector field in K^r, independent of K^r. Then there is an $\epsilon_0 > 0$ with the following property. For every ϵ, $0 < \epsilon < \epsilon_0$, if

$$\phi_{v,\epsilon}(K) = \text{all } p + \epsilon v(p), \qquad p \in K, \tag{9.1}$$

then $K \cap \phi_{v,\epsilon}(K) = 0$. Now for any chain A^r in K, $\phi_{v,\epsilon}A$ is a chain in $\phi_{v,\epsilon}(K)$. For cycles A^r, we define

$$\mathfrak{L}(A, v) = LC(\phi_{v,\epsilon}A, A). \tag{9.2}$$

Because of (9.1) it is clear that the result is independent of the choice of ϵ. The definition will be extended in §14.

The following lemma gives the relation to the previous $\mathfrak{L}$.

LEMMA 8. *Let M^n be a partial orientable manifold, and let f be a semi-regular mapping of M^n into E^{2n-1}. Let $v(p)$ be defined in $f(\partial M)$, and independent of $f(\partial M)$, and point into $f(M)$ at each p. Then*

$$\mathfrak{L}_f(M) = \mathfrak{L}(f\partial M, v). \tag{9.3}$$

To prove the lemma, we note that for ϵ sufficiently small, the set of all $\phi_{v,\epsilon'}(\partial M)$ for $0 \leqq \epsilon' \leqq \epsilon$ projects in a one-one manner into a subset M' of M, and M' contains all of M within some neighborhood of ∂M. Setting $M^* = M - M'$ defines a chain M^* to which Lemma 5 applies. Clearly

$$\mathfrak{L}_f(M) = LC(f\partial M^*, f\partial M) = LC(\phi_{v,\epsilon}\partial M, f\partial M) = \mathfrak{L}(f\partial M, v).$$

Let f be an imbedding of the r-manifold M^r in E^{2r+1}, and let $v(p)$ be defined so that $v(p)$ is independent of $f(M)$ at $f(p)$. If deformations f_t and v_t are given so that each f_t is an imbedding and each v_t is independent of $f_t(M)$, clearly $\mathfrak{L}(fM, v)$ is constant. The next lemma (used in the proof of Theorem 4) generalizes this.

LEMMA 9. *Let f_t be a regular deformation of M^r in E^{2r+1} such that f_0 and f_1 are imbeddings. Let v_t be a continuous vector field independent of $f_t(M)$ for each t. Then*

$$\mathfrak{L}(f_1M, v_1) = \mathfrak{L}(f_0M, v_0) \qquad \text{for } r \text{ even},$$

$$\mathfrak{L}(f_1M, v_1) \equiv \mathfrak{L}(f_0M, v_0) \pmod 2 \qquad \text{for } r \text{ odd}.$$

REMARK. The lemma clearly holds if M^r is a cycle in a complex K^n, f_t being a regular deformation in each closed $\bar{\sigma}_i^n$. The crossings (see below) may be taken interior to n-cells $\sigma_i^n = U_i$ of K.

It is easily seen by the methods in [1] that a slight alteration of f_t for $0 < t < 1$ will give a new f_t with the following property. If, for a certain t_1, f_{t_1} is not one-one, say $q_0 = f_{t_1}(p_1) = f_{t_1}(p_2)$, then this is the only double point, and the portions U_1 and U_2 of M near p_1 and p_2 are crossing each other as t moves through t_1. That is, if $u_{i1}, \cdots, u_{ir}$ are independent vectors tangent to $f_{t_1}(U_i)$ at q $(i = 1, 2)$, and

$$u' = \frac{\partial f_t(p_1)}{\partial t}\bigg|_{t=t_1} - \frac{\partial f_t(p_2)}{\partial t}\bigg|_{t=t_1},$$

then these $2r + 1$ vectors are independent.

Take t' and t'' very close to t_1, with $t' < t_1 < t''$. Since $\mathfrak{L}$ is constant over intervals containing no such t_1, it is sufficient to prove that the relations hold with $f_{t'}$ and $f_{t''}$ replacing f_0 and f_1. We may clearly accomplish the deformation f_t in two steps: first, push $f_{t'}(U_1)$ in the direction of u' so that it crosses $f_{t'}(U_2)$ (pushing $f_{t'}(p_1)$ a distance $(t'' - t')u'$); second, deform the result into $f_{t''}(M)$. We may replace v_t by $v_{t'} = v$ in all this if t' and t'' are close enough together. We now need merely prove the relations for the mappings before and after the first deformation, which we call g_t, using $0 \leqq t \leqq 1$.

Set

$$M_t = g_t(M), \qquad N_t = \phi_{v,\epsilon}(M_t) \qquad (0 \leqq t \leqq 1),$$

with a small $\epsilon > 0$. Except in U_1, these are independent of t. They define singular chains A and B such that (with M oriented)

$$M_1 = M_0 + \partial A, \qquad N_1 = N_0 + \partial B.$$

We find

$$\begin{aligned}\Delta &= \mathfrak{L}(M_1, v) - \mathfrak{L}(M_0, v) = LC(N_1, M_1) - LC(N_0, M_0)\\ &= LC(\partial B, M_1) + LC(N_0, M_1 - M_0)\\ &= LC(\partial B, M_1) + (-1)^{r+1}LC(\partial A, N_0)\\ &= KI(B, M_1) + (-1)^{r+1}KI(A, N_0).\end{aligned}$$

Since $u_{11}, \cdots, u_{1r}, u'$ are independent, we may suppose $A \cap B = 0$. Also g_0 is one-one in $M - U_2$. Hence we may clearly suppose

$$KI(B, g_1(M - U_2)) = KI(A, \phi_{v,\epsilon} g_0(M - U_2)) = 0.$$

Therefore

$$\Delta = KI(B, g_1U_2) + (-1)^{r+1}KI(A, \phi_{v,\epsilon} g_0U_2) = [1 + (-1)^{r+1}]KI(A, g_0U_2),$$

which proves the lemma.

II. The immersion theorem

10. Some deformations related to certain vector fields

We first show when one vector field in $M^r \subset E^{2r+1}$ may be deformed into another one. In Lemma 11 we show how the boundary of a partial manifold may be moved over to a desired position, and in Lemma 12, we show how the boundary may be twisted to point in given directions.

Lemma 10. *Let $M^r \subset E^{2r+1}$ $(r \geqq 1)$ be a connected closed orientable manifold, and let v_0 and v_1 be continuous vector fields in M, each independent of M. Then there is a deformation $v_t (0 \leqq t \leqq 1)$ of v_0 into v_1 so that each v_t is independent of M, if and only if $\mathfrak{L}(M, v_0) = \mathfrak{L}(M, v_1)$.*

Remark. If the normal bundle of M^r in E^{2r+1} is simple,[2] the proof is easy to give.

The necessity of the condition in the theorem is clear; we shall prove the sufficiency. We first deform v_0 and v_1 into fields of unit normal vectors. Next, let K be a simplicial complex forming a fine enough subdivision of M so that each cell of K is nearly flat, and so that v_0 is nearly constant in each cell. Now as considerations of dimensionality show at once, we may deform v_1 so that $v_1 = v_0$ in K^{r-1}. We now suppose v_0 and v_1 are of this nature.

Let $\{\sigma_i^r\}$ denote the similarly oriented r-cells of K. For each $p \in M$, let $S(p)$ denote the unit r-sphere about p whose plane is normal to M at p, and let $S'(p) \subset S(p)$ denote the subsphere orthogonal to $v_0(p)$. Let S_0^r denote a fixed r-sphere, and q_0, a fixed point of it. For each σ_i^r we introduce a "coordinate system"

[2] See [3]. The present lemma belongs properly in the subject considered there. Some of the details omitted in the present proof may be found there. In the present paper we need only the case that M is a sphere. By cutting it into two cells and using the theorem that any sphere-bundle over a cell is simple, the proof could be materially simplified in this case.

into the $S(p)$ as follows. For each $p \in \bar{\sigma}_i^r$ and each $q \in S_0^r$, $\xi_i(p, q)$ is a point of $S(p)$; for each p, it is an orthogonal (distance preserving) mapping, and this mapping is continuous in p; furthermore, $\xi_i(p, q_0) = v_0(p)$. (Since the σ_i^r are nearly flat, it is easy to construct ξ_i first over the part S_0^{r-1} of S_0^r orthogonal to q_0, so as to map into the $S'(p)$; it is then uniquely extendable over S_0^r.) Let

$$\xi_i^{-1}(p, q') = q \quad \text{if} \quad q' = \xi_i(p, q).$$

If we orient M and S_0^r, orient the $S(p)$ so that the orientations of M and $S(p)$ at p determine the positive orientation of E, and choose the ξ_i so that they are rotations (i.e. sense-preserving), then each $\xi_i^{-1}(p, \xi_j(p, q))$ for each p will be a rotation.

Set

$$\psi_i(p) = \xi_i^{-1}(p, v_1(p)). \tag{10.1}$$

Since $v_1(p) = v_0(p)$ in $\partial\sigma_i^r$, $\psi_i(p) = q_0$ there. Hence ψ_i maps σ_i^r into S_0^r so that $\partial\sigma_i^r$ goes into q_0, and thus ψ_i has a degree d_i over σ_i^r. Set

$$X(v_1) = \sum d_i \sigma_i^r. \tag{10.2}$$

Since $\dim(K) = r$, this is a cocycle.

Suppose v_1 is deformed as follows. Take any σ_j^{r-1}; $\psi_i(p) = q_0$ here. As t runs from 0 to 1, let $\psi_{it}(\sigma_j^{r-1})$ sweep over S_0^r with the degree α_j, keeping $\psi_{it}(\partial\sigma_j^{r-1}) = q_0$ and $\psi_{i1}(\sigma_j^{r-1}) = q_0$. (Thus if I is the unit interval $0 \leqq t \leqq 1$, and $\Psi_i(t, p) = \psi_{it}(p)$, Ψ_i maps $I \times \sigma_j^{r-1}$ into S_0^r with the degree α_j.) We may extend[3] ψ_{it} over the rest of M, requiring that it be independent of t except in the cells of $St(\sigma_j^{r-1})$. Set

$$v_t'(p) = \xi_i(p, \psi_{it}(p)), \qquad p \in \bar{\sigma}_i^r, \text{ each } i;$$

then $v_0'(p) = v_1(p)$, and each v_t' is a field of unit normal vectors. Let d_i' denote the degree defined with the help of v_1'. Then clearly for any σ_i^r,

$$d_i' = d_i + [\sigma_j^{r-1} : \sigma_j^r]\alpha_j,$$

and hence

$$X(v_1') = X(v_1) + \alpha_j \delta\sigma_j^{r-1}.$$

Since we may carry out this process for each σ_j^{r-1} in turn, we may alter X by any coboundary.

For a small $\epsilon > 0$, if $\omega_k(\sigma_i^r)$ denotes the cell σ_i^r displaced in the direction of v_k a distance ϵ (this mapping need not be one-one), then

$$\mathfrak{L}(M, v_k) = LC(\sum \omega_k \sigma_i^r, M) \qquad (k = 1, 2). \tag{10.3}$$

We shall show that

$$LC(\omega_1\sigma_i^r - \omega_0\sigma_i^r, M) = d_i. \tag{10.4}$$

[3] See ALEXANDROFF-HOPF, *Topologie I*, Berlin, 1936, p. 501 Hilfsatz Ia.

This quantity is defined, since $v_0 = v_1$ in $\partial\sigma_i^r$. Let us flatten σ_i^r into σ', lying in a space E^r. A slight alteration of v_0 will make it constant in $\bar{\sigma}_i^r$. Let E^{r+1} be a plane through a point $p_0 \in \sigma_i^r$, orthogonal to E^r, and consider S_0^r as the unit sphere in E^{r+1} about p_0. If we project the chain $\omega_1\sigma_i^r - \omega_0\sigma_i^r$ parallel to E^r into E^{r+1}, and then in E^{r+1} away from p_0 into S_0^r, $\omega_0\sigma_i^r$ will go into a point, say q_0, and $\omega_1\sigma_i^r$ will go into a chain $\omega'\sigma_i^r$. Now

$$LC(\omega_1\sigma_i^r - \omega_0\sigma_i^r, E^r) = LC(\omega'\sigma_i^r, E^r).$$

This also equals $LC(\omega'\sigma_i^r, p_0)$, considering this as defined in E^{r+1} (which is oriented like $S(p_0)$). We may suppose the ξ_i chosen so that after the above alterations and projection, $v_1(p) + p$ becomes the point $\psi_i(p)$. Hence

$$LC(\omega'\sigma_i^r, E^r) = d_i.$$

Interpreting the looping coefficients as Kronecker indices shows that the looping coefficients with E^r are the same as with M. Thus (10.4) is proved.

Adding the equations (10.4) gives

$$\begin{aligned}\sum d_i &= LC(\sum \omega_1\sigma_j^r - \sum \omega_0\sigma_i^r, M)\\ &= \mathfrak{L}(M, v_1) - \mathfrak{L}(M, v_0) = 0.\end{aligned}$$

Since M is closed, connected and oriented, there is a one-one correspondence h between the cohomology classes of dimension r of M and the integers, given by

$$h(\sum \alpha_i\sigma_i^r) = M \cdot \sum \alpha_i\sigma_i^r = \sum \alpha_i.$$

It follows that

$$h(X(v_1)) = \sum d_i = 0, \qquad X \backsim 0.$$

Consequently we may deform v_1 into v_1' so that $X(v_1') = 0$, i.e. $d_i' = 0$ for each i. Now by a theorem of Hopf,[4] we may deform ψ_i' into q_0 in each $\bar{\sigma}_i^r$, keeping it at q_0 in $\partial\bar{\sigma}_i^r$. This defines a corresponding deformation of v_1' into v_0 in $\bar{\sigma}_i^r$, keeping it fixed in $\partial\sigma_i^r$. Thus v_1 is deformed into v_0 in M, and the lemma is proved.

LEMMA 11. *Let f_0 and f_1 be imbeddings of the manifold M^r in E^v. Let $L(p)$ be the segment $f_0(p)f_1(p)$. Let no two of these have common points. For each $p \in M$ let there be a plane $T(p) = T^{v-r}(p)$ in E^v such that*

(a) *$f_0(p)$ and $f_1(p)$ are in $T(p)$,*

(b) *$T(p)$ has only $f_i(p)$ in common with the tangent plane to $f_i(M)$ at $f_i(p)$ ($i = 0, 1$),*

(c) *The function $T(p)$ is smooth (Compare* [1], §24).

Then there is a smooth deformation ϕ_t of E^v ($0 \leqq t \leqq 1$) such that

(d) *each ϕ_t is an imbedding, and ϕ_0 is the identity,*

(e) *$\phi_1(f_0(p)) = f_1(p)$ ($p \in M$),*

(f) *for a given neighborhood U of the set of all segments $L(p)$, $\phi_t(p) = p$ for $p \in E^v - U$ and $0 \leqq t \leqq 1$.*

[4] See ALEXANDROFF-HOPF, loc. cit., p. 504, Satz III_n, or H. WHITNEY, Duke Math. J., vol. 3 (1937), pp. 46–50, Appendix.

REMARKS. Any segment $L(p)$ may reduce to a single point $f_0(p) = f_1(p)$. If M is a partial manifold, and $f_0 = f_1$, together with first partial derivatives, in ∂M, the proof below holds, and each ϕ_t is the identity, together with first partial derivatives, at all points of ∂M. The most important application of the lemma is to the case $M^r = \partial M^n$; a given mapping f_0 of M^n is then altered to f_1, so that f_1 is a given mapping in ∂M. If the mappings are of class C^r, we may make each ϕ_t of class C^r.

Take any p_0 in M. By (a), (b) and (c), it is easy to see that for some neighborhood U_0 of p_0 and some $\alpha_0 > 0$, the set of points q in planes $T(p)$ with $p \in U_0$ which are within a distance α_0 of $L(p)$ fills out a neighborhood of $L(p_0)$ in E^ν in a smooth one-one way (compare the proof of [1], Lemma 21). Hence, since the $L(p)$ are distinct, there is a positive continuous function $\alpha(p)$ (or a constant $\alpha > 0$ if M is compact) such that if $R(p)$ is the set of points of $T(p)$ within a distance $\alpha(p)$ of $L(p)$, then the $R(p)$ fill out a neighborhood of $\sum L(p)$ in a smooth one-one-way, and $\sum R(p) \subset U$. We may choose a smooth function $\eta(p) > 0$ such that if $L'(p)$ is the segment $L(p)$ extended in each direction by the amount $\eta(p)$, and $C(p)$ is the cylinder (of dimension $\nu - r$) in $T(p)$ with axis $L'(p)$ and of radius $\eta(p)$, then $C(p) \subset R(p)$ for $p \in M$ (see [1], Lemma 25). It is easy to set up an expression depending smoothly on $\eta(p)$ and the (locally oriented) length of $L(p)$, which defines a smooth deformation of $R(p)$ into itself with the properties that it is constant in $R(p) - C(p)$, carries $f_0(p)$ into $f_1(p)$, and is an imbedding for each t. Letting this define ϕ_t in $\sum R(p)$ and setting $\phi_t(p) = p$ in $E^\nu - \sum R(p)$ proves the lemma.

LEMMA 12. *Let $M^n (n \geqq 2)$ be a partial manifold, let ∂M be a closed manifold, and let f be a mapping of class C^2 of M into E^{2n-1} such that in some neighborhood U of ∂M, f is an imbedding. Let $u(p)$ be a smooth vector field in ∂M, pointing into M at p in ∂M; set*

$$v(p) = \nabla f(u, p), \qquad p \in \partial M.$$

Let $v'(p)$ be a smooth vector field in $f(\partial M)$, independent of $f(\partial M)$, such that

$$\mathfrak{L}(f\partial M, v') = \mathfrak{L}(f\partial M, v).$$

Then there is a smooth mapping f' of M into E such that $f' = f$ in $M - U$, f' is an imbedding in U, $f' = f$ in ∂M, f' is arbitrarily close to f (but not together with first derivatives) in M, and

$$\nabla f'(u, p) = v'(p), \qquad p \in \partial M.$$

REMARKS. The assumption that ∂M is closed could be easily removed. A more accurate statement about the class of f and of f' could be given, but we shall not need it.

Since we need define f' in U only, we may consider U as lying in E^{2n-1}, and let f be the identity; then $v(p) = u(p)$. Set

$$p_t^* = p + tv(p), \qquad 0 \leqq t \leqq 1.$$

This is a smooth mapping of $I \times \partial M$ into E^{2n-1}. (Since f is of class C^2, v and p_t^* are of class C^1 in terms of the original coordinate systems in M.) For some $t_0 > 0$, this is an imbedding for the values $0 \leqq t \leqq t_0$, since u is smooth. For t_0 small enough, we may project p_t^* into U. Say p_t^* projects into p_t. Now the points of U near ∂M are uniquely expressible in the form p_t ($p \in \partial M$, $0 \leqq t \leqq t_0$), and

$$\left.\frac{\partial p_t}{\partial t}\right|_{t=0} = v(p).$$

By Lemma 10, there is a deformation $v_t'(p)$ ($0 \leqq t \leqq 1$) of $v'(p) = v_0'(p)$ into $v(p) = v_1'(p)$, such that each v_t' is independent of ∂M. We may replace $v_t'(p)$ by a smooth function $v_t^*(p)$ as follows. First set

$$v_t''(p) = v_0'(p) \qquad (t \leqq \tfrac{1}{3}), \qquad v_t''(p) = v_1'(p) \qquad (t \geqq \tfrac{2}{3}),$$

$$v_t''(p) = v_{t'}'(p) \qquad (t' = 3(t - \tfrac{1}{3}),\ \tfrac{1}{3} \leqq t \leqq \tfrac{2}{3}).$$

Then $v_t''(p)$ is smooth except for $\frac{1}{3} \leqq t \leqq \frac{2}{3}$. Now approximate to $v_t''(p)$ by a smooth vector function $v_t^*(p)$ for $\frac{1}{6} \leqq t \leqq \frac{5}{6}$, the approximation being closer and closer, together with first partial derivatives, as $t \to \frac{1}{6}$ or $t \to \frac{5}{6}$. (See [1], Theorem 2, (a) and (d). We could either make use of Theorem III of the author's paper in Trans. Am. Math. Soc., vol. 36 (1934), pp. 63–89, using first derivatives for $t < \frac{1}{3}$ and $t > \frac{2}{3}$, or note simply that in the approximation in Lemma 6, loc. cit., with $m = 0$, the first partial derivatives have automatically the desired approximation property.) Setting $v_t^*(p) = v_t''(p)$ for $t \leqq \frac{1}{6}$ and $t \geqq \frac{5}{6}$ makes v_t^* smooth for $0 \leqq t \leqq 1$ (see [1], Lemma 10). Moreover, with a close enough approximation, $v_t^*(p)$ is independent of ∂M for $0 \leqq t \leqq 1$.

Set

$$v_t(p) = \frac{\partial p_t}{\partial t}.$$

For certain numbers α and β to be determined later, with $0 < \alpha < \beta < t_0$, set

$$p_t' = p + \int_0^t v_{s/\alpha}^*(p)\, ds \qquad (0 \leqq t \leqq \alpha), \tag{10.5}$$

$$p_t' = p + \int_0^\alpha v_{s/\alpha}^*(p)\, ds + \int_0^{t-\alpha} v_s(p)\, ds \qquad (\alpha \leqq t \leqq \beta). \tag{10.6}$$

Cover ∂M with a finite set of coordinate systems $\{x_i\}$. Let V be the maximum of $|\, v_t^*(p)\,|$, $|\, v_t(p)\,|$, $|\, \partial v_t^*(p)/\partial x_i\,|$, $|\, \partial v_t(p)/\partial x_i\,|$. Now

$$\frac{\partial p_t'}{\partial t} = v_{t/\alpha}^*(p) \qquad (t \leqq \alpha), \qquad \frac{\partial p_t'}{\partial t} = v_{t-\alpha}(p) \qquad (t \geqq \alpha);$$

since $v_1^*(p) = v_1'(p) = v(p) = v_0(p)$, the mapping thus defined is smooth. Also, since

$$\frac{\partial p_t'}{\partial x_i} = \frac{\partial p}{\partial x_i} + \int_0^t \frac{\partial v_{s/\alpha}^*(p)}{\partial x_i}\, ds \qquad (t \leqq \alpha),$$

and similarly for $t \geqq \alpha$, we find

$$\left|\frac{\partial p_t'}{\partial x_i} - \frac{\partial p}{\partial x_i}\right| \leqq V\beta \qquad (0 \leqq t \leqq \beta).$$

Since the $\partial p/\partial x_i$ are independent for each p, and $\partial p_t'/\partial t$ is independent of them, by choosing β small enough we may insure that the $\partial p_t'/\partial x_i$ and $\partial p_t'/\partial t$ are independent for each p and t; hence the mapping is regular. Moreover, since

$$p_t = p + \int_0^t v_s(p)\, ds,$$

we have

$$p_\beta' - p_\beta = \int_0^\alpha v^*_{s/\alpha}(p)\, ds - \int_{\beta-\alpha}^{\beta} v_s(p)\, ds,$$

and

$$|\, p_\beta' - p_\beta \,| \leqq 2V\alpha, \qquad \left|\frac{\partial p_\beta'}{\partial x_i} - \frac{\partial p_\beta}{\partial x_i}\right| \leqq 2V\alpha,$$

$$\left|\frac{\partial p_t'}{\partial t} - \frac{\partial p_t}{\partial t}\right|_\beta = |\, v_{\beta-\alpha}(p) - v_\beta(p) \,|\,.$$

Hence, keeping β fixed, we may choose α so small that the mapping p_t' at $t = \beta$ is arbitrarily close to that of p_t at β, together with first derivatives.

Set

$$\phi(t, p) = p_t' - p_t \qquad (t \leqq \beta), \qquad\qquad \phi(t, p) = 0 \qquad (t \geqq t_0).$$

This is a mapping of the part of U outside $\beta < t < t_0$, which we have just seen may be taken arbitrarily small, with first derivatives. Hence, by [5], there is a smooth extension of ϕ through $\beta \leqq t \leqq t_0$, which may be taken arbitrarily small, together with first derivatives. Setting

$$p_t' = p_t + \phi(t, p) \qquad (\beta < t < t_0)$$

completes the definition of f' in U. By making ϕ and its first derivatives small enough, we insure that we have a close approximation to f, and that the new mapping is regular. Since

$$\left.\frac{\partial p_t'}{\partial t}\right|_{t=0} = v_0^*(p) = v_0'(p) = v'(p),$$

we have $\nabla f'(u(p), p) = v'(p)$, completing the proof.

11. A twisted cube

We wish to show how, for n even, an n-cube in E^{2n-1} can be slightly altered in position so that, on one face, there will be a "double twist".

THEOREM 5. *Let M_0 be an n-cube in $E^n \subset E^{2n-1}$, n even and $\geqq 2$, and let N_0 be one of its faces. Then there is an immersion f of M_0 in E^{2n-1} with the following properties:*

(1) *f is arbitrarily near the identity Θ.*

(2) *$f = \Theta$ in ∂M_0.*

(3) *$f = \Theta$, together with first derivatives, in $\partial M_0 - N_0$.*

(4) *$\mathfrak{L}_f(M_0) = 2$ or -2 at will.*

It would be easy to make f of class C^∞.

That it is possible to have $\mathfrak{L}_f(M_0) \neq 0$ can be seen at once as follows. Take the mapping of an $(n-1)$-cube into E^{2n-2} with just one self-intersection as defined in the preceding paper; translating E^{2n-2} in E^{2n-1} gives a mapping f of M_0 into E^{2n-1} with a line of self-intersections, and with ∂M_0 intersecting itself in two points. Make slight deformations so as to remove the self-intersections of ∂M_0. Since n is even, it is easily seen that pulling ∂M_0 away from itself at one of these points in opposite directions has the opposite effect on $\mathfrak{L}_f(M_0)$; hence we may obtain $\mathfrak{L}_f(M_0) \neq 0$.

We must show how a mapping may be obtained to have also the remaining properties. We shall first describe geometrically the case $n = 2$. Take a long rectangle of paper, carry the right hand end up, towards the left, down (cutting through itself), and to the right again; it will then be approximately in its original position, except for the presence of a somewhat cylinder shaped portion near the middle. This mapping may be defined by

$$y_1 = x_1 - \frac{2x_1}{1 + x_1^2}, \qquad y_2 = \frac{1}{1 + x_1^2}, \qquad y_3 = x_2, \tag{11.1}$$

the y_1-axis pointing East (to the right), the y_2-axis up, and the y_3-axis South. By pulling the right half fairly taut, and a little to one side, the cylindrical piece is made very narrow, and is pulled to a sharp angle, say to about 12° from the direction from left to right. This renders the two long edges nearly straight again. (If a thin strip is cut off one of the long edges, it is found to have no self-intersections, and may be formed from a straight strip by simply twisting one end.)

The two long edges are now made into straight lines by a slight distortion of 3-space. A contraction in one direction turns the edge (now a rectangle) into a square. The resulting mapping has all the required properties except that there is a twist along two edges instead of along only one. Let us round off the corners. We could now either curl over all the right hand edge (see the proof below), or greatly contract the lower and right hand portions, pulling one twisted part of the edge all the way around the right hand end to a position near the other twisted part (see the figure). The figure shows all these operations except for the straightening of the wavy edge.

We turn now to the general case. By analogy with the above, we shall take the self-intersection defined in the preceding paper, for E^{n-1} mapped into E^{2n-2}, and translate E^{2n-2} in E^{2n-1} but moving it at an angle $\theta = \tan^{-1}(1/10)$ instead of $\pi/2$.

Set

$$u' = (1 + x_1^2) \cdots (1 + x_{n-1}^2),$$

$$y_1 = x_1 - \frac{2x_1}{u'}, \qquad y_i = x_i \qquad (i = 2, \cdots, n-1),$$

(11.2)

$$y_n = \frac{1}{u'}, \qquad y_{n+i-1} = \frac{x_1 x_i}{u'} \qquad (i = 2, \cdots, n-1),$$

$$y_{2n-1} = x_n + 10x_1 .$$

For x_n fixed, we obtain the mapping referred to. Since that mapping is regular, and x_n appears in y_{2n-1} only, the present mapping is regular. Let us call it f_0. The self-intersections are:

(11.3) $$f_0(1, 0, \cdots, 0, \alpha - 10) = f_0(-1, 0, \cdots, 0, \alpha + 10).$$

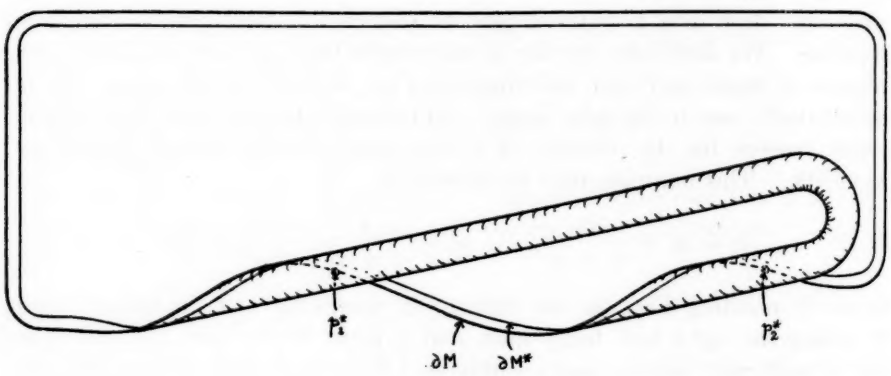

FIG. 2

Let M_1 be the part of E^n defined by $|x_n| \leqq 100$. Consider the affine mapping f' of M_1 into E^{2n-1} defined by omitting the terms in (11.2) containing u':

(11.4) $$y_i = x_i, \qquad y_{n+i-1} = 0 \qquad (i = 1, \cdots, n-1),$$

$$y_{2n-1} = x_n + 10\, x_1 .$$

Then f_0 is close to f', together with first partial derivatives, except near the x_n-axis. (Taking $n = 2$, the reader is advised to plot the parallelogram $y_1 = x_1$, $y_3 = x_2 + 3x_1$, for $|x_1| \leqq 3$ and $|x_2| \leqq 9$.)

Our next object is to replace the mapping f_0 by a mapping f_2 with the same kind of self-intersections, and such that for any $p = (x_1, \cdots, x_n)$, and some a,

(11.5) $$f_2(p) = f'(p) \quad \text{if} \quad |x_n| = 100 \quad \text{or} \quad \sum_{i=1}^{n-1} x_i^2 \geqq a^2.$$

We shall do this by defining a deformation ϕ_t of E^{2n-1}, and setting

$$f_t(p) = \phi_t(f_0(p)) \qquad (0 \leqq t \leqq 2). \tag{11.6}$$

Let N_1^+ and N_1^- be the parts of ∂M_1 with $x_n = 100$ and $x_n = -100$ respectively. For each $q \in E^{2n-1}$, let $T(q)$ be the n-plane containing q which is parallel to the axes of $y_1, y_n, y_{n+1}, \cdots, y_{2n-2}$. We shall define ϕ_t in two parts. First, ϕ_t $(0 \leqq t \leqq 1)$ will carry each point $q = f_0(p)$ near $f'(\partial M_1)$ in the plane $T(q)$ into $f'(p)$, and will be the identity in E^{2n-1} outside a neighborhood of $f'(\partial M_1)$. Then $\phi_1(q)$ is near q if q is far from the y_{2n-1}-axis. Consequently it is easy to define ϕ_t $(1 \leqq t \leqq 2)$ so as to make (11.5) hold.

First, note that

$$\lambda(u) = \lambda(u; a, b) = 1 - 3\left(\frac{u-a}{b-a}\right)^2 + 2\left(\frac{u-a}{b-a}\right)^3 \qquad (a \leqq u \leqq b)$$

has the properties

$$\lambda(a) = 1, \qquad \lambda'(a) = \lambda(b) = \lambda'(b) = 0;$$

hence, if $\lambda = 1$ for $x < a$ and $\lambda = 0$ for $x > b$, λ is smooth. The maximum derivative of $-\lambda$ is at $x = \frac{1}{2}(a+b)$, and has the value $3/[2(b-a)]$.

For each $q = (y_1, \cdots, y_{2n-1})$ in E^{2-1} there is a unique point

$$\begin{aligned} \pi^+(q) &= (x_1, y_2, \cdots, y_{n-1}, 0, \cdots, 0, y_{2n-1}) \qquad \left(x_1 = \frac{y_{2n-1} - 100}{10}\right) \\ &= f'(x_1, y_2, \cdots, y_{n-1}, 100) \end{aligned} \tag{11.7}$$

in $f'(N_1^+) \cap T(q)$. Also,

$$\sigma^+(q) = f_0(x_1, y_2, \cdots, y_{n-1}, 100) \tag{11.8}$$

is in $T(q)$, and is clearly the only point of $f_0(N_1^+)$ in $T(q)$. Set

$$\rho^+(q) = |\sigma^+(q) - q|,$$

and

$$\phi_t(q) = q + t\lambda(\rho^+(q); 1, 9)[\pi^+(q) - \sigma^+(q)], \qquad \rho^+(q) \leqq 9. \tag{11.9}$$

Thus for any $q_0 = f_0(p_0) \in f_0(N_1^+)$ and any q in $T(q_0)$ within a distance 1 of q_0, ϕ_1 moves q by that vector which carries $f_0(p_0)$ into $f'(p_0)$; if q is at a distance 9 from q_0, then $\phi_t(q) = q$.

Set $\omega = y_{2n-1} - 10y_1$, and define the half-spaces

$$E^+{:}\omega > 0, \qquad E^-{:}\omega < 0.$$

Since $\omega = 100$ at points of $f'(N_1^+)$, and $\max [2x_1/u'] = 1$, $\omega \geqq 99$ at points of $f_0(N_1^+)$. Hence if $\rho^+(q) \leqq 9$, $\omega(q) \geqq 9$, and $q \in E^+$. We may therefore, using N_1^- in place of N_1^+, define $\pi^-(q)$ etc., and define $\phi_t(q)$ for $\rho^-(q) \leqq 9$ so as to have corresponding properties. Setting $\phi_t(q) = q$ in the rest of E^{2n-1} completes the definition of ϕ_t for $t \leqq 1$.

By direct substitution, we find that for any $p = (x_1, \cdots, x_n)$, taking $x_n > 0$,

$$f_1(p) - f_0(p) = \lambda[\rho^+(p); 1, 9]v(p),$$

$$v(p) = \left(\frac{2x_1'}{u''}, 0, \cdots, 0, -\frac{1}{u''}, -\frac{x_1' x_2}{u''}, \cdots, -\frac{x_1' x_{n-1}}{u''}, 0\right),$$

$$x_1' = x_1 + \frac{x_n - 100}{10}, \qquad u'' = (1 + x_1'^2)(1 + x_2^2) \cdots (1 + x_{n-1}^2).$$

Hence, for any $p \in M_1$ with sufficiently large $x_1^2 + \cdots + x_{n-1}^2$, $f_1(p)$ is close to $f_0(p)$ and hence to $f'(p)$, together with first partial derivatives. Consequently, for some a, if $q = f_1(p)$ is at least a distance a from the y_{2n-1}-axis, and hence $y_1^2 + \cdots + y_{n-1}^2$ is large, then $x_1^2 + \cdots + x_{n-1}^2$ is large, and the above statement holds. For such values of q, set

$$w(q) = f'(q) - f_1(q); \tag{11.10}$$

Then w and its first derivatives are small if q is at least a distance a from the y_{2n-1}-axis; it vanishes in $f'(\partial M_1) = f_1(\partial M_1)$. It follows that w may be extended over E^{2n-1} so that it is small everywhere, together with first partial derivatives, and vanishes in $f'(\partial M_1)$. (This fact may be proved as follows. By a contraction in each $(2n-1)$-plane $y_{2n-1} - 10y_1 = \alpha$, we may bring the set A_a of points distant at least a from the y_{2n-1}-axis into A_1; then w is carried into w_a say, defined in $[f'(M_1) \cap A_1] \cup f'(\partial M_1)$. By taking a large enough, we may make w_a and its first partial derivatives arbitrarily small. We now apply the theorem of [5]—the fact that A_1 is not bounded is clearly inconsequential,—and reverse the above contraction.) We now set

$$\phi_t(q) = q + (t-1)w(q) \qquad (1 \leqq t \leqq 2). \tag{11.11}$$

Then each ϕ_t is an imbedding, and (11.5) holds.

Consider E^n as a subspace of E^{2n-1}. If we define the affine mapping of E^{2n-1}

$$\psi\colon\ y_i' = y_i \ (i = 1, \cdots, 2n-2), \qquad y_{2n-1}' = y_{2n-1} - 10y_1,$$

this carries f_2 into a mapping f_3, where

$$f_3(p) = \psi(f_2(p)) \qquad (p \in M_1)$$

such that, by (11.5),

$$f_3(p) = p \quad \text{if} \quad |x_n| = 10 \quad \text{or} \quad \sum_{i=1}^{n-1} x_i^2 \geqq a^2. \tag{11.12}$$

Next we shall change f_3 to f_4 so that $f_4 =$ identity, together with first derivatives, at all points of N_1^-. Let M_2 be a partial manifold contained in M_1 and containing all points of M_1 with $x_1^2 + \cdots + x_{n-1}^2 \leqq a^2$. For instance, let M_2 be the set of all points of E^n whose distance from the $(n-1)$-cell $x_n = 0, x_1^2 + \cdots + x_{n-1}^2 \leqq a^2$, is at most 100. Then $f_3(p) = p$ in ∂M_2. Let $u(p)$ be the inward normal at $p \in \partial M_2$. It is carried into a vector field $v(p)$ in E^{2n-1} by f_3. Let

$v'(p)$ be defined in ∂M_2 and lie in E^{2n-1}, let it equal $v(p)$ in $\partial M_2 - N_1^+ \cup N_1^-$, and let it be the inward normal in $\partial M_2 \cap N_1^-$; define it in $\partial M_2 \cap N_1^+$ so that

$$\mathfrak{L}(\partial M_2, v') = \mathfrak{L}(\partial M_2, v). \tag{11.13}$$

Let f_4 be the mapping given by Lemma 12. Then $f_4(p) = p$ in ∂M_2, and setting $f_4(p) = p$ in $M_1 - M_2$ gives the required mapping f_4. We may let $f_4(p) = p$ if $\sum_{i=1}^{n-1} x_i^2 \geqq a^2$.

Now contract E^{2n-1} with a factor 100 in the y_{2n-1}-direction, and with a factor $b \geqq a$ in the other directions. This carries f_4 into a mapping

$$f(p) = \theta(f_4(\theta^{-1}(p))).$$

A certain rectangular parallelopiped M_b' lying in M_1 and containing M_2 is carried into the cube

$$M_0\colon\ |x_i| \leqq 1 \qquad (i = 1, \cdots, n).$$

Since $f_4(p) = p$ in $M_1 - M_a'$ and f_4 leaves y_{2n-1} unchanged, choosing b large enough makes f arbitrarily near the identity Θ in M_0. Clearly $f(p) = p$ in ∂M_0, and $\partial f(p)/\partial x_i = \partial\Theta(p)/\partial x_i$ in $\partial M_0 - N_0$, where $N_0 = \partial M_0 \cap \theta(N_1^+)$. There remains to prove (4) of the theorem.

Since a reflection in E^{2n-1} will cause a change in sign in $\mathfrak{L}$, it is sufficient to show that $\mathfrak{L}_f(M_0) = \pm 2$. Clearly

$$\mathfrak{L}_f(M_0) = \mathfrak{L}_{f_4}(M_b') = \mathfrak{L}_{f_4}(M_2).$$

Since $u(p)$ points into M_2, and f_4 carries $u(p)$ into $v'(p)$ while f_3 carries $u(p)$ into $v(p)$, Lemma 8 and (11.13) give

$$\mathfrak{L}_{f_4}(M_2) = \mathfrak{L}(\partial M_2, v') = \mathfrak{L}(\partial M_2, v) = \mathfrak{L}_{f_3}(M_2).$$

Since the changes from f_0 to f_2 and to f_3 in M_1, and hence in M_2, are obtainable by deformations of E^{2n-1}, which leave Kronecker indices unchanged,

$$\mathfrak{L}_{f_2}(M_2) = \mathfrak{L}_{f_0}(M_2).$$

Hence there remains to prove

$$\mathfrak{L}_{f_0}(M_2) = KI(f_0 M_2^*, f_0 \partial M_2) = \pm 2. \tag{11.14}$$

The intersections of $f_0(\partial M_2)$ with $f_0(M_2)$ are:

$$p_1 = (-1, 0, \cdots, 0, 100), \qquad p_1^* = (1, 0, \cdots, 0, 80),$$

$$p_2 = (\ 1, 0, \cdots, 0, -100), \qquad p_2 = (-1, 0, \cdots, 0, -80),$$

$$p_i \in \partial M_2, \qquad p_i^* \in M_2^*, \qquad f_0(p_i) = f_0(p_i^*).$$

Let A and B be the parts of M_2 satisfying:

$$A\colon\ -2 \leqq x_1 \leqq -\tfrac{1}{2}, \qquad B_2\colon\ \tfrac{1}{2} \leqq x_1 \leqq 2,$$

respectively; we let these be chains, oriented like M_2. Then A and B contain neighborhoods of p_2^* and p_1^* in M_2^* respectively, and ∂A and ∂B contain neighborhoods of p_1 and p_2 in ∂M_2 respectively. Since $f_0(\partial A)$ and $f_0(\partial B)$ do not intersect B and A respectively at other points, and n is even, we have

$$\begin{aligned} KI(f_0M_2^*, f_0\partial M_2) &= KI(f_0B, f_0\partial A) + KI(f_0A, f_0\partial B) \\ &= KI(f_0B, f_0\partial A) + (-1)^n KI(f_0\partial A, f_0B) \\ &= [1 + (-1)^n]KI(f_0B, f_0\partial A) = \pm 2, \end{aligned}$$

completing the proof.

12. Proof of the immersion theorem

We can now prove the following theorem, with the help of Theorem 2; recall that that theorem is proved with the exception of the case M non-orientable, n odd.

THEOREM 6. *Given any n-manifold or partial manifold M ($n \geqq 2$) of class C^r ($r \geqq 1$ finite or infinite), and a continuous mapping f of M into E^{2n-1}, there is an immersion g of M arbitrarily close to f, of class C^r.*

REMARK. By Lemma 1, we may make the mapping completely semi-regular.

We suppose M is connected; otherwise, we would apply the theorem to each component of M separately. It is sufficient to find an immersion g of class C^1; for a sufficiently close approximation to g by a mapping g' of class C^r is automatically an immersion also. We may suppose that M is of class C^{16}; if it were not, we could take a manifold or partial manifold M^* of class C^{16}, and an imbedding ϕ of M onto M^*, set $f^*(q) = f(\phi^{-1}(q))$ ($q \in M^*$), find an immersion g^* of M^*, and set $g(p) = g^*(\phi(p))$ ($p \in M$). Finally, by Lemma 1, we may suppose that f is of class C^{16} and semi-regular. We must now divide the proof into four cases.

CASE I. M is closed, and n is odd. By Theorem 3, we may name the singular points $p_1, p_1', \cdots, p_s, p_s'$ in such a manner that p_i and p_i' are of opposite types. If we show how to approximate to f by a function g' in which p_1 and p_1' are no longer singular points, and with no new singular points, a repetition of this process gives a function g approximating to f and without singular points; g is then an immersion.

The method of proof is as follows. Join p_1 to p_1' by an arc A in M. (This may be taken as an arc along which f has a self-intersection.) A neighborhood of A in M may be expressed as the image of a sphere plus interior q_0^n; f then gives a smooth mapping f^* of Q_0^n into E^{2n-1}, with two singular points q_1 and q_1', of opposite types. If we approximate to f^* by an immersion g^*, which agrees with f^*, together with first partial derivatives, in ∂Q_0^n, this gives the required g' in M.

We shall describe the construction of g^* in the case $n = 2$, in spite of the fact that 2 is not odd. By Lemma 2, we may suppose that f^* near q_1 and q_1' is as shown in Fig. 1. We may choose the mapping of Q_0^n into M so that near $f^*(q_1)$

(and similarly, near $f^*(q_1')$) $f^*(Q_0^n)$ goes up to the top line $x_1 = 0$ in the figure, and down again a short distance, to $x_1 = -\beta$. If we cut off small pieces R and R' of Q_0^n containing q_1 and q_1' respectively, so that the new figure ends at $x_1 = \beta$ near these points, then it is pretty clear that by pushing part of $x_1 = \beta$ into part of $x_1 = -\beta$, we can change f^* to an immersion h', so that $h'(\partial(Q_0^n - R \cup R')) = f^*(\partial Q_0^n)$. Since Q_0^n is easily mapped into $Q_0^n - R \cup R'$, we obtain h with $h(q) = f^*(q)$, $q \in \partial Q_0^n$.

Noting that the part $\partial' R$ of ∂R inside Q_0^n must, under f^*, curl over the top edge in Fig. 1, we see that watching the image of the vector normal to ∂Q_0^n, under f^* and under h', as we go along $\partial R - \partial' R$ and along $\partial' R$, along the latter we obtain one complete twist more than along the former. The same is true near q_1'. As a result, since the singular points are of opposite types, $\mathfrak{L}_h(Q_0^n) = \mathfrak{L}_{f^*}(Q_0^n)$. Therefore we may apply Lemma 12, and obtain from h a mapping g^* which, like h, is an immersion, but agrees with f^*, together with first derivatives, in ∂Q_0^n.

We turn now to the proof for Case I. Turning to §22, (c), we see that given p_1, p_1' may be chosen so that these two points are the ends of arcs A and A', forming a smooth curve in M, and both mapping into a single arc in E^{2n-1}. (We could take, for A, any smooth arc in M which ends like the above arc at p_1 and at p_1'.) Choose coordinate systems about p_1 and p_1' as in Lemma 2; they are of class C^2. With the proper choice of the x_1-direction in each case, the ends of A are given by

$$0 \leqq x_1 \leqq c, \qquad x_2 = \cdots = x_n = 0. \tag{12.1}$$

It is easy to define independent smooth vector functions $v_i(p)$ $(i = 2, \cdots, n)$ along A so that $v_i(p_1)$ and $\pm v_i(p_1')$ are in the x_i-directions at p_1 and p_1' respectively. By considering M as lying in E^{2n+1}, and projecting the points $p + \sum \alpha_i v_i(p)$ into M, we may define a coordinate system throughout a neighborhood of A. We may extend the system beyond p_1 and p_1'. We have now a neighborhood of A expressed as the imbedding of a cylinder in M, and the ends of the cylinder are mapped approximately into the sets defined by

$$x_1 = -\beta, \qquad x_2^2 + \cdots + x_n^2 \leqq \alpha^2. \tag{12.2}$$

(Note that x_1 may be replaced by $-x_1$ in Lemma 2.) If we alter the two ends of the cylinder slightly so that they coincide with $x_1 = -\beta$, and then round off the two edges (which are near $x_1 = -\beta$, $x_2^2 + \cdots + x_n^2 = \alpha^2$), we obtain a partial manifold Q_0^n, expressible as a sphere plus interior.

Let ψ be the imbedding of Q_0^n into M; set

$$f^*(q) = f(\psi(q)) \qquad (q \in Q_0^n),$$

$$q_1 = \psi^{-1}(p_1), \qquad q_1' = \psi^{-1}(p_1').$$

We may clearly suppose that f^* is an imbedding in some neighborhood of ∂Q_0^n. We now consider $(x_1, \cdots, x_n)$ as coordinates in Q_0^n near q_1 and near q_1'; then the

sets defined by (12.2) with α replaced by $\alpha' = \alpha/2$ are parts N_0 and N_0' of ∂Q_0^n. Let λ be a real-valued function of class C^2 such that

$$\lambda(t) = 1 \text{ if } |t| \leqq \alpha'/2, \qquad \lambda(t) = 0 \text{ if } |t| \geqq \alpha'.$$

Let R and R' be the parts of Q_0^n about q_1 and q_1' respectively, defined by

$$-\beta \leqq x_1 \leqq -\beta + 2\beta\lambda[(x_2^2 + \cdots + x_n^2)^{1/2}]. \tag{12.3}$$

Then if $R_0 = \overline{Q_0^n - R \cup R'}$, ∂R_0 is a manifold of class C^1, and f^* immerses R_0 in E^{2n-1}. For each $q \in E^{2n-1}$, let $T_1(q)$ be the n-plane through q parallel to the axes of $y_1, y_{n+1}, y_{n+2}, \cdots, y_{2n-1}$. If $T(p) = T_1(f^*(p))$, $p \in N_0$, these planes cut $f^*(N_0)$ in the manner prescribed in Lemma 11. Moreover, if

$$N_1 = \partial R - N_0, \qquad N_1' = \partial R' - N_0',$$

then each $T(p)$ ($p \in N_0$) cuts $f^*(N_1)$ in exactly one point $f_1^*(p)$. Pushing N_0 and N_0' onto N_1 and N_1' clearly defines an imbedding θ of Q_0^n onto R_0 such that $f_1^*(p) = f^*(\theta(p))$ agrees with the above f_1^* in N_0 and N_0'. Now f_1^* and f^*, defined in $N_0 - \partial N_0$, satisfy the conditions for f_0 and f_1 in Lemma 11; define ϕ_t by that lemma, and set

$$h_1(p) = \phi_1(f_1^*(p)), \qquad p \in Q_0^n.$$

Carry out a similar deformation of E^{2n-1} about $f_1^*(R')$, forming the mapping h of Q_0^n in E^{2n-1}. Then (see the remarks following the lemma) h is of class C^2, and $h(p) = f^*(p)$ ($p \in \partial Q_0^n$). Since f^* is an immersion of R_0, f_1^* and h are immersions of Q_0^n. By taking β small enough, we may make h an arbitrarily good approximation to f^* (not of course with first derivatives).

Let $u(p)$ be the inward normal vector at p in ∂Q_0^n. If we replace h by g^*, using Lemma 12, so that

$$g^*(p) = f^*(p), \qquad \nabla g^*(u(p), p) = \nabla f^*(u(p), p) \qquad (p \in \partial Q_0^n),$$

then $g^* = f^*$, together with first partial derivatives, in ∂Q_0^n; hence, if

$$g'(p) = g^*(\psi^{-1}(p)) \quad (p \in \psi(Q_0^n)), \qquad g'(p) = f(p) \quad \text{otherwise},$$

g' will be smooth in M. Since h and hence g^* (see Lemma 12) is an immersion in Q_0^n, g' has no singular points in $\psi(Q_0^n)$, and the proof for Case I will be complete.

To apply Lemma 12, we need merely prove

$$\mathfrak{L}(h\partial Q_0^n, \nabla h(u)) = \mathfrak{L}(h\partial Q_0^n, \nabla f^*(u)). \tag{12.4}$$

By Theorem 2 and Lemma 8, these numbers are the algebraic number of singular points of h and f^* in Q_0^n. It is 0 for h, since h is regular. It is also 0 for f^*, since f^* has just two singular points, q_1 and q_1', and these are of opposite types.

Case II. M is closed and n is even. By Theorem 3, the number of singular points is even; call them $p_1, p_1', \cdots, p_s, p_s'$. If we proceed as under Case I, the only difficulty is at the last step; the two numbers in (12.4) may differ from

each other by any even integer $2k$. Set $M^* = \psi(Q_0^n)$. Choose k points $r_1, \cdots, r_k$ in ∂M^*. For each i, we may express a small piece M_i' of $f(\overline{M - M^*})$ about r_i as the imbedding ϕ_i of an n-cube M_i, one face N_i going into part of $f(\partial M^*)$. If the M_i' are small, and we take $M_i \subset E^{2n-1}$, we may in an obvious fashion extend ϕ_i to be an imbedding of a neighborhood U_i of M_i in E^{2n-1} into E^{2n-1}. Let θ_i be the mapping of M_i into E^{2n-1} given by Theorem 5, with θ_i = identity in ∂M_i, together with first partial derivatives in $\partial M_i - N_i$. Then

$$F(p) = \phi_i\theta_i\phi_i^{-1}f(p) \quad (p \in f^{-1}(M_i')), \quad F(p) = f(p) \quad \text{otherwise},$$

defines a smooth mapping of $M - M^*$, agreeing with f except in the $f^{-1}(M_i')$. Choosing the correct sign in (4) of Theorem 5 in each case, we obtain

$$\mathfrak{L}(h\partial Q_0^n, \nabla h(u)) = \mathfrak{L}(h\partial Q_0^n, \nabla F^*(u)), \tag{12.5}$$

where F^* and its first partial derivatives are defined at points of ∂Q_0^n in Q_0^n in terms of F with the help of the imbedding ψ. We may now apply Lemma 12 as before.

Case III. M is open. Choose compact partial manifolds M_i in M by Lemma 20, Appendix. We shall define mappings $f_0 = f, f_1, f_2, \cdots$ with f_i arbitrarily close to f_{i-1}, such that f_i is regular in M_i, and $f_i = f_{i-1}$ in M_{i-1}. Then $g = \lim f_i$ exists and is an immersion.

Suppose f_{i-1} is properly defined. The number of singular points of f_{i-1} in M_i is finite; none are in M_{i-1}. It is sufficient to show how an arbitrarily slight alteration of f_{i-1} will get rid of one of these, say p_1. By (c) of Lemma 20, we may join p_1 to a point p_2 in $M - M_i$ by an arc $A \subset M - M_{i-1}$; we may clearly keep A away from ∂M. Take a small neighborhood U of p_2, and express a neighborhood of $f_{i-1}(U)$ in E^{2n-1} as the image ϕ of part of E^{2n-1}, so that for $E^n \subset E^{2n-1}$, $\phi(E^n)$ contains $f_{i-1}(U)$. Using the last mapping of §4, we may then alter f_{i-1} in U so that it has two singular points, say p_1' and p_1''. If n is odd, then by Theorem 2, these are of opposite types; hence one of these, say p_1', is of opposite type to that of p_1. Applying the proof in Case I, we alter f_{i-1} in a neighborhood of A, together with U, getting rid of the singularities at p_1 and p_1'. If n is even, we apply the proof in Case II, using p_1 and either of p_1', p_1''.

Case IV. M is compact but not closed; then $\partial M \neq 0$. Add a small piece onto M along part of ∂M, and remove a closed n-cell from the new portion, obtaining a new open manifold M', with $M \subset M'$, and extend f through M'. By Case III, we may alter f to give an immersion g of M'; this defines an immersion g of M, and completes the proof of the Theorem.

13. Further immersion theorems

We consider here what may be done with ∂M in an immersion of M.

Theorem 7. *Let f be a smooth mapping of the connected partial manifold M^n into E^{2n-1} ($n \geqq 2$) which is an imbedding in a neighborhood of ∂M. If M is not compact, there exists an immersion g of M in E^{2n-1} which is arbitrarily close to f*

and equals f in a neighborhood of ∂M. If M is compact and n is odd [is even], g exists if and only if $\mathfrak{L}_f(M)$ is $= 0$ [is $\equiv 0 \bmod 2$].

Using Lemma 1, we first replace f by a semi-regular mapping, which we again call f. If M is not compact, we may clearly apply the proof in Case III* of the last theorem. If M is compact, the proof in Case I or in Case II applies. That the condition is necessary is a consequence of Theorem 2.

THEOREM 8. *Any compact partial manifold M may be immersed in E^{2n-1} so that the mapping f is an imbedding in a neighborhood of ∂M, and*

$$f(\partial M) \cap f(M - \partial M) = 0.$$

We shall not discuss the case of open manifolds. The theorem being clear if $n = 2$ (all M^2 being known), we assume $n \geqq 3$; also we may suppose M is connected. Let f_0 be the immersion given by Theorem 6; by Lemma 1, we may suppose f_0 is completely semi-regular. By §22, Appendix, the intersections of $f_0(\partial M)$ with $f_0(M - \partial M)$ are on ends of arcs as described in (e_3) and (e_4). Since the number of such arcs is finite, it will be sufficient to show how to get rid of an intersection of either kind.

Consider first an intersection as in (e_4). Let A and B be the arcs of M with $f_0(A) = f_0(B)$, and let U be a neighborhood of A in M. It is easy to define a smooth imbedding g of M in itself which is the identity outside U, and squeezes the whole arc A up into a part of U beyond the end point of A which is in $M - \partial M$, so that

$$g(M) \cap A = 0.$$

(Use a coordinate system about A, as in the proof for Case I of Theorem 6.) If we choose U so small that

$$f_0(U - A) \cap f_0(M - U) = 0,$$

and set $f(p) = f_0(g(p))$, we will clearly have removed the arc of self-intersection without introducing any further intersections.

Now take the case (e_3). Say $f_0(A) = f_0(B)$, all of A being in $M - \partial M$. Since $n \geqq 3$, we may extend A in one direction, forming a smooth arc A' with one end in ∂M, so that

$$f_0(A' - A) \cap f_0(M - A) = 0.$$

We now define g and f as above, with A' in place of A.

III. FURTHER INTERSECTION THEORY

14. Looping coefficients of vector fields in complexes in space

Using the definition of §9, we shall discuss "looping coefficients" of vector fields and pairs of vector fields with pairs of cycles in a smooth (not necessarily finite) complex $K^r \subset E^{2r+1}$. We derive two formulas which are useful, in particular, in studying $\mathfrak{L}_f(M)$ for a partial manifold $M^n, f(M) \subset E^{2n-1}, n = r + 1$.

In the rest of Part III, we study the situation when the above cycles are replaced by chains.

For any r-chain A of K, define the r-chain $\phi_{v,\epsilon}A$ as in §9. *Assuming v independent of K*, we shall take ϵ (or a positive continuous function $\epsilon = \epsilon(p)$ if K is infinite) so small that

$$\phi_{v,\alpha}(K) \cap \phi_{v,\beta}(K) = 0 \quad \text{if} \quad -\epsilon < \alpha < \beta < \epsilon. \tag{14.1}$$

Generalizing (9.2), set (for finite r-cycles A, B)

$$\mathfrak{L}(A, B, v) = LC(\phi_{v,\epsilon}A, B). \tag{14.2}$$

If we cut a closed manifold M^n into two parts M_1 and M_2, and $v(p)$ points into $f(M_1)$ at $f(p)$ in $f(\partial M_1)$ (where $f(M) \subset E^{2n-1}$), then the following lemma, with $r = n - 1$, $A = B = f\partial M_1$, relates $\mathfrak{L}_f(M_1)$ to $\mathfrak{L}_f(M_2)$. The lemma will be generalized in (19.10).

LEMMA 13. *Take K and v as above. Then for finite r-cycles A and B,*

$$\mathfrak{L}(A, B, v) = (-1)^{r+1}\mathfrak{L}(B, A, -v). \tag{14.3}$$

By (14.1), $B = \phi_{v,0}B$ may be deformed into $\phi_{v,-\epsilon}B$ without touching $\phi_{v,\epsilon}A$, $\phi_{v,\epsilon}A$ may be deformed into A without touching $\phi_{v,-\epsilon}B$. Hence

$$\begin{aligned} LC(\phi_{v,\epsilon}A, B) &= (-1)^{r^2+1}LC(B, \phi_{v,\epsilon}A) \\ &= (-1)^{r+1}LC(\phi_{v,-\epsilon}B, \phi_{v,\epsilon}A) = (-1)^{r+1}LC(\phi_{-v,\epsilon}B, A), \end{aligned}$$

which proves the lemma.

REMARK. Let α, β, γ be oriented arcs in E^3 joining the points p and q. Set $A = \alpha - \beta$, $B = \beta - \gamma$. Then it is easy to define v so that

$$\mathfrak{L}(A, B, v) = 1, \qquad \mathfrak{L}(B, A, v) = 0.$$

Suppose there is a small "fin" stretching out from K, in the direction of a vector field $u(p)$ (independent of K). Then $\mathfrak{L}(A, B, v)$, or $\mathfrak{L}_f(M)$ etc., may be determined by studying the intersection of $\phi_{v,\epsilon}A$ etc. with this fin. To show this, note first that by deforming $\phi_{v,\alpha}A$ into $\phi_{v,\beta}A$, we define a chain $\psi_{v,\alpha,\beta}A$ such that

$$\partial\psi_{v,\alpha,\beta}A = \phi_{v,\beta}A - \phi_{v,\alpha}A - \psi_{v,\alpha,\beta}\partial A. \tag{14.4}$$

Now let $u(p)$ be a continuous vector function in K, independent of K, such that K, u, v are independent in K^{r-1}. That is, for each σ^r, face σ^{r-1}, and $p \in \bar{\sigma}^{r-1}$, the 2-plane through $u(p)$ and $v(p)$ has only p in common with the tangent plane to $\bar{\sigma}^r$ at p. We may suppose u, v are unit vector functions. Then we may take any η, $0 < \eta < \epsilon$, and define

$$\mathfrak{L}(A, B, u, v) = KI(\phi_{u,\eta}A, \psi_{v,0,\epsilon}B). \tag{14.5}$$

Note that if $\partial A = \partial B = 0$, then this is defined without the restriction that K, u and v be independent in K^{r-1}.

This quantity is reducible to the former:

LEMMA 14. *If* $\partial A = \partial B = 0$, *then*

$$\mathfrak{L}(A, B, u, v) = \mathfrak{L}(B, A, v) + (-1)^r \mathfrak{L}(A, B, u). \tag{14.6}$$

In particular,

$$\begin{aligned} \mathfrak{L}(A, A, v, v) &= 2\mathfrak{L}(A, A, v) && (r \text{ even}), \\ &= 0 && (r \text{ odd}). \end{aligned} \tag{14.7}$$

By the method of proof of the last lemma, we find

$$\begin{aligned} \mathfrak{L}(A, B, u, v) &= KI(\psi_{v,0,\epsilon}B, \phi_{u,\eta}A) \\ &= LC(\phi_{v,\epsilon}B - B, \phi_{u,\eta}A) = LC(\phi_{v,\epsilon}B, A) - LC(B, \phi_{u,\eta}A) \\ &= LC(\phi_{v,\epsilon}B, A) + (-1)^r LC(\phi_{u,\eta}A, B), \end{aligned}$$

which gives (14.6).

15. The type of complex we shall use[5]

In the rest of Part III we shall use only complexes that are simplicial, or at least have certain properties of simplicial complexes. In particular:

(a_1) Each closed cell $\bar{\sigma}^n$ may be represented as the (smooth) one-one image $\theta(\bar{\sigma}_0^n)$ of a convex closed cell in E^n.

(a_2) Each closed r-face $\bar{\sigma}^r$ of σ^n is the intersection of $n - r$ $(n-1)$-faces $\bar{\sigma}_1^{n-1}, \cdots, \bar{\sigma}_{n-r}^{n-1}$ of $\bar{\sigma}^n$.

(a_3) For $p \in \sigma^r$, the tangent planes to the $\bar{\sigma}_i^{n-1}$ there have only the tangent plane to $\bar{\sigma}^r$ in common.

It will be convenient to call the *tangent cone* $\Gamma(\bar{\sigma}^r, p)$ of $\bar{\sigma}^r$ at $p \in \bar{\sigma}^r$ the set of all vectors v tangent to $\bar{\sigma}^r$ at p; i.e. the set of all possible limits $\lim [\phi(t) - p]/t$, $\phi(0) = p$, $\phi(t) \in \bar{\sigma}^r$. The *tangent space* $\bar{\Gamma}(\bar{\sigma}^r, p)$ is the set of all linear combinations of vectors of $\Gamma(\bar{\sigma}^r, p)$. We say a vector of $\bar{\Gamma}$ is *parallel* to $\bar{\sigma}^r$ at p. In terms of this, (a_3) is equivalent to

$$\bar{\Gamma}(\bar{\sigma}^r, p) = \bar{\Gamma}(\bar{\sigma}_1^{n-1}, p) \cap \cdots \cap \bar{\Gamma}(\bar{\sigma}_{n-r}^{n-1}, p) \qquad (p \in \bar{\sigma}^r). \tag{15.1}$$

A cell may be in the form of a cube for example. Note that we may not subdivide a proper face of a cell without subdividing the cell itself, for then (a_3) would be contradicted.

Given $p \in K$, let $\Gamma(K, p)$ denote the set of all tangent cones of closed cells containing p, at p. These form parts of linear spaces which are unrelated except for those corresponding to incident cells. It is not so easy to give meaning to $\bar{\Gamma}(K, p)$.

Suppose f is a smooth mapping of K into E^r. If f is regular in the part of a closed cell $\bar{\sigma}$ near $p \in \bar{\sigma}$, then

$$\Gamma(f(\bar{\sigma}), f(p)) = \nabla f \Gamma(\bar{\sigma}, p).$$

This is true with K in place of $\bar{\sigma}$; now $\Gamma(f(K), q)$ is formed of cones lying in E^r.

[5] The author expects to give a more general theory of this subject in a paper on "Complexes of manifolds."

We shall say f is an *immersion* of K if it is regular (in each closed cell) and proper, and is an *imbedding* if, in addition, it is one-one, and further, ∇f is one-one in each $\Gamma(K, p)$. In the latter case, we say K is a complex *in* the second space.

EXAMPLE. No complex may be imbedded onto a pair of tangent circles.

LEMMA 15. *Any complex K^n may be imbedded in E^{2n+1} so that it has no limit set.*

It is not difficult to define an imbedding cell by cell, by the methods in [1]; of course it is easy and standard if K is simplicial. For the last statement (trivial if K is finite), compare [1], p. 665, footnote 32.

We shall let σ_i^r denote the cells of K, and set $\tau_i^r = f(\sigma_i^r)$. If K lies in E^ν, we may let τ_i^r denote its cells, thinking of f as the identity mapping.

LEMMA 16. *Let K lie in E^ν. Then each $\bar\tau_i^r$ may be enclosed in a larger cell $\tau_i'^r$, of the same class C^γ as K, such that if τ_j^s is a face of τ_i^r, then $\tau_j'^s$ lies in $\tau_i'^r$.*

REMARK. The lemma extends in an obvious way to the case, if $K = K^n$, that f is an imbedding in a neighborhood of each $\partial\sigma_i^n$.

Of course we take $\tau_i'^0 = \tau_i^0$. Suppose the $\tau_i'^1, \cdots, \tau_i'^{s-1}$ have been constructed. Take any $\bar\tau_i^s$, the image $\theta(\bar\tau_0^s)$ of a convex cell $\bar\tau_0^s$ in E^s. It is easy to define extensions of the boundary cells of τ_0^s, and define θ over these, mapping into the extended faces $\tau_j'^t$ of τ_i^s. Because of (a_3), we may now extend θ over a neighborhood of $\bar\tau_i^s$. In a sufficiently small neighborhood, which may be taken as a convex cell, θ is an imbedding.

16. On general position of a complex and vector field in space

Let f be a smooth mapping of a complex K (see §15) into E^ν. We say f is in *general position* if it is proper, and:

(b_1) For each s, f is regular in K^s at all points of $K^{\nu-s}$.

(b_2) For each s, each $p \in K^s$ and each $q \in K^{\nu-s-1}$, if $p \neq q$ then $f(p) \neq f(q)$.

(b_3) If $u_1 \in \Gamma(K^s, p)$, $u_2 \in \Gamma(K^{\nu-s}, p)$, $u_1 \not\equiv u_2$, then $\nabla f(u_1, p) \not\equiv \nabla f(u_2, p)$.

One could combine (b_1) and (b_3) in a more complicated statement. Note that (b_3) uses Γ, not $\bar\Gamma$. Any imbedding of K^n into E^ν, $\nu \geqq 2n$, is clearly a mapping in general position. From the above we deduce:

(b_1') f is regular in K^s if $\nu \geqq 2s$.

(b_2') f is one-one in K^s if $\nu \geqq 2s + 1$.

(b_3') ∇f is one-one in $\Gamma(K^s, p)$ if $\nu \geqq 2s$.

Let $(y_1, \cdots, y_\nu)$ be a coordinate system in E^ν, and let π be the projection: $\pi(y_1, \cdots, y_{\nu-1}, y_\nu) = (y_1, \cdots, y_{\nu-1})$ of E^ν into $E^{\nu-1}$.

Say f is in *general position with reference to the y_ν-direction* if the above holds, and in addition, πf is in general position in $E^{\nu-1}$. We could obviously generalize this, using a set of independent directions.

Let v be a continuous vector field, defined in a closed subcomplex of K, and with values in E^ν. Then $\nabla\pi v$ has values in $E^{\nu-1}$. We say f and v are in *general position* if f is, and also:

(b_4) For each s, $v(p)$ is independent of any $f(\bar\sigma^s)$ at $f(p)$ for all points $p \in K^{\nu-s-1}$.

(b_5) For each s, $v(p)$ is independent of $\nabla f[\bar\Gamma(\bar\tau_i^s, p) + \bar\Gamma(\bar\tau_j^{\nu-s-1}, p)]$, where defined.

Letting $\sigma^{\nu-s-1}$ be a face of σ^s shows that (b_4) is a consequence of (b_5). We shall often omit (b_5).

Finally, f and v are in *general position with reference to the y_ν-direction* if the above holds also for πf and $\nabla \pi v$.

LEMMA 17. *Let K be of class C^2, let f map K into E^ν with no limit set, and let $v_1, v_2, \cdots$ be smooth vector functions, each defined in a subcomplex of K and with values in E^ν. Let f and each v_k be in general position. Then by an arbitrarily small rotation of the axes we may make this hold with reference to the y_ν-direction. The lemma holds if we omit (b_5) in the hypothesis and conclusion.*

REMARKS. If K is not of class C^2, or the v_k are not smooth, we may use a C^1-homeomorphic K' which is of class C^2, and smooth v_k' approximating to the v_k. The application will be to the case that K_1 is a subdivision of a smooth manifold M^n, $K = K_1^{n-1}$, and the v_k are independent of the $(n-1)$-cells of K and tangent to the n-cells of K. We could use several independent directions in place of the single y_ν-direction. If $K = K^r$, we could allow the limit set L_f of f to exist, provided it is of zero $(\nu - r - 1)$-extent.

EXAMPLE. Let K be a subdivision of an open arc, and let f map K into E^3 so that it winds like a ball of string, having a 2-sphere as limit set. Then f is proper, but no projection into E^2 is proper.

To prove the lemma, let $S^{\nu-1}$ be the unit sphere in E^ν; its points may be thought of as directions in E^ν. Let R_{s1} be the set of all those directions parallel to $f(K^s)$ at a point of $f(K^{\nu-s-1})$. Let R_{s2} be all those through points $f(p)$ and $f(q)$ with $p \in K^s$, $q \in K^{\nu-s-2}$, $p \neq q$, (which implies $f(p) \neq f(q)$). Let R_{s3} be all directions of vectors $u' = \nabla f(u, p)$, $u = u_1 - u_2$, $u_1 \in \Gamma(K^s, p)$, $u_2 \in \Gamma(K^{\nu-s-1}, p)$, $u_1 \neq u_2$ (which implies $u' \neq 0$). Let R_{s4} be all those defined by $v_k(p)$ plus a vector parallel to some $f(\bar{\sigma}^s)$ at $p \in K^{\nu-s-2}$. If we are using (b_5), let R_{s5} be all those defined by $v_k(p) + u_1' + u_2'$, $u_i' = \nabla f(u_i)$, $u_1 \in \bar{\Gamma}(\bar{\tau}_i^s, p)$, $u_2 \in \bar{\Gamma}(\bar{\tau}_j^{\nu-s-2}, p)$.

Since the directions parallel to $f(K^s)$ at a point p form a set of finite $(s-1)$-extent in $S^{\nu-1}$, and this set varies smoothly with p, R_{s1} is a finite or denumerable sum of sets of finite $(\nu - 2)$-extent.[6] In R_{s2}, since p and q range over sets of finite s- and $(\nu - s - 2)$-extent respectively, R_{s2} is a (denumerable) sum of sets of finite $(\nu - 2)$-extent. In R_{s3}, take any p in any σ^t, and suppose σ^t is a face of σ^s and of $\sigma^{\nu-s-1}$. If $u_1 \in \Gamma(\bar{\sigma}^s, p)$ and $u_2 \in \Gamma(\bar{\sigma}^{\nu-s-1}, p)$, these range over sets of finite s- and $(\nu - s - 1)$-extent respectively. But adding a vector of $\Gamma(\bar{\sigma}^t, p)$ to each leaves their difference unchanged; hence $u_1 - u_2$ ranges over sets of finite $(\nu - 1 - t)$-extent. Letting p vary shows that the directions vary over a sum of sets of finite $(\nu - 2)$-extent. Since the directions defined by $v_k(p) + \nabla f(u, p)$, $u \in \bar{\Gamma}(\bar{\sigma}^s, p)$, form a sum of sets of finite s-extent, R_{s4} is as required. In R_{s5}, $v_k(p) + u_1' + u_2'$, p fixed in σ^t, defines directions of finite $(\nu - 2 - t)$-extent; hence R_{s5} is as required.

Consequently by [1], Lemmas 13 and 14, $R = \sum R_{si}$ has no inner points in

[6] See [1], in particular, Lemma 15. It would be possible to use dimension instead of extent.

S^{r-1}, so that arbitrarily near the y_r-direction there is a direction not in R. Rotate axes so as to make this direction the new y_r-direction. Then since any vector not in this direction projects into a non-zero vector, it is easily seen that f and each v_k are in general position with reference to the new y_r-direction, completing the proof.

LEMMA 18. *Any smooth complex K^n may be mapped into any E^r so that it is in general position with reference to a given direction.*

REMARK. This is a generalization of the imbedding and immersion theorems of [1] for manifolds.

By the remark to the last lemma, we may suppose that K is of class C^2. Imbed K^n in E^μ, $\mu = \max[\nu, 2n+1]$, so that it has no limit set (Lemma 15). It is then in general position. By Lemma 17, applying a small rotation makes it in general position with reference to any chosen direction. If $\mu = \nu$, we let this be the given direction. If $\mu > \nu$, we choose any direction; projecting in this direction into $E^{\mu-1}$ gives a mapping in general position in $E^{\mu-1}$. Repeat the process till we reach E^r.

17. The fins and corresponding projections

Take a smooth complex $K = K^r \subset E^{2r+1}$, and let K and v be in general position with reference to the y_{2r+1}-direction (using the identity mapping), omitting (b_5). We shall suppose K and v are of class C^2; if this is not so, and we do not wish to change K or v, we could replace the normal planes $T(q)$ below by a smooth function $T(q)$; the properties given will then hold.

Let $\{\tau_i^s\}$ denote the cells of K. Each τ_i^r may be enclosed in a larger cell τ_i' of class C^2 (Lemma 16). Suppose v is defined over $\bar{\tau}_i^r$; extend it to be of class C^2 over τ_i'. With small enough τ_i', we will still have general position. Set

$$\phi_{v,t}(p) = p + tv(p) \qquad (p \in \tau_i'). \tag{17.1}$$

For any point set $R \subset \tau_i'$, let $\Phi_v(R, t)$ denote all points $\phi_{v,t'}(p)$ with $p \in R$ and with $0 \leqq t' \leqq t$; let $\Phi_v^*(R, t)$ denote the same, with $0 < t' \leqq t$. We call these the *fin* and *deleted fin* respectively of τ_i', v and t. Since K and v are of class C^2, t_i' may be chosen so that the *double fin* $\Phi_v(\tau_i', t_i') \cup \Phi_{-v}(\tau_i', t_i')$ is expressed by ϕ as an imbedding of the product of τ_i' and the interval $-t_i' \leqq t \leqq t_i'$. Moreover, for each q in the double fin, if $T_0(q)$ is the set of all points in the normal plane $T(q)$ to the fin at q which are at a distance $\leqq t_i'$ from q, then these fill out a closed neighborhood $W_v(\tau_i', t_i')$ of the interior of the double fin in a one-one way, being again the imbedding of a product. If we set

$$P_v(q') = q \quad \text{if} \quad q' \in T_0(q), \tag{17.2}$$

this is a smooth projection of $W_v(\tau_i', t_i')$ onto the double fin. Finally, let $\Lambda_v(R, t)$ denote all points $q' \in T(q)$, with $q \in \Phi_v^*(p, t)$, $p \in R$, such that $|q' - q| \leqq t|q - p|$.

Since K is in general position with reference to the y_{2r+1}-direction, omitting (b_5), the (b_k) give

(b_1^*) π is regular in K.
(b_2^*) If $p \in K^{r-1}$, $q \in K$, $p \neq q$, then $\pi p \neq \pi q$.
(b_3^*) Distinct vectors in K at any point map under π into distinct vectors.
(b_4^*) $\nabla \pi v(p)$ is independent of any $\pi \bar{\sigma}^s (s \leqq r)$ at p for $p \in K^{r-1}$.
(b_6^*) There is a neighborhood V^* of K^{r-1} in K such that if $p \in V^*$, $q \in K$, $p \neq q$, then $\pi p \neq \pi q$.

To prove (b_6^*), we use (b_1^*) and (b_3^*) to show that π is one-one in a neighborhood of any point of K^{r-1}, use (b_2^*) to show that π is one-one in K^{r-1}, and apply Lemma 25, §24.

Let $v_1, v_2, \cdots$ be vector functions in K with the same properties as v, such that at most a finite number are defined in any τ_i^r, and for each v_k, some v_l equals $-v_k$. For any R, let $N_t R$ denote all points whose distance from R is $\leqq t$. We shall choose numbers $t_i > 0$ such that the following properties hold, for any τ_i^r and v_k defined in τ_i^r.

(c_1) $\Lambda_{v_k}(\tau_i' \cap N_{t_i}\tau_i^r, t_i) \subset W_{v_k}(\tau_i', t_i')$.
(c_2) $\Phi^*_{v_k}(\tau_i^r, t_i) \cap \Phi_{-v_k}(\tau_i^r, t_i) = 0$.
(c_3) $\pi\Phi^*_{v_k}(\tau_i^r, t_i) \cap \pi\Phi_{-v_k}(\partial\tau_i^r, t_i) = 0$.
(c_4) $\Lambda_{v_k}(\tau_i' \cap N_{t_i}\tau_i^r, t_i) \cap \tau_i^r = 0$.
(c_5) $\pi\Lambda_{v_k}(\tau_i' \cap N_{t_i}\tau_i^r, t_i) \cap \pi\partial\tau_i^r = 0$.
(c_6) $\pi\Lambda_{v_k}(\tau_i' \cap N_{t_i}\partial\tau_i^r, t_i) \cap \pi\tau_i^r = 0$.

It is sufficient to prove each property (c_i) separately. We shall write Λ in place of Λ_{v_k}. We may use a single k, since but a finite number of v_k are defined in τ_i^r. (c_1), (c_2) and (c_4) are clear. The proof of (c_3) is essentially contained in that of (c_5), so we turn to (c_5) and (c_6).

Take $p \in \partial\tau_i^r$. For a small enough t_1, if $q' \in \Lambda(\tau_i' \cap N_{t_1}p, t_1)$, say $q' \in T_0(q)$, $q \in \Phi^*_{v_k}(p')$, $p' \in \tau_i' \cap N_{t_1}p$, then $q - p' = \alpha v_k(p')$, and $|q' - q| \leqq t_1 |q - p'|$, so that the angle between $p'q'$ and $v_k(p)$ is small; also $p' - p$ is approximately a vector $u_1 \in \Gamma(\bar{\tau}_i^r, p)$; furthermore, for any $p'' \in \tau_i^r \cap N_{t_1}p$, $p'' - p$ is approximately a vector $u_2 \in \Gamma(\bar{\tau}_i^r, p)$; therefore, by ($b_4^*$), we may suppose that

$$\nabla\pi[(q' - p') + (p' - p)] \neq \nabla\pi(p'' - p), \qquad \pi q' \neq \pi p''.$$

This gives

$$\pi\Lambda(\tau_i' \cap N_{t_1}p, t_1) \cap \pi(\tau_i^r \cap N_{t_1}p) = 0.$$

Since $\tau_i^r - N_{t_1}p$ is compact, because of (b_2^*) we may clearly take $t_2 \leqq t_1$ so that

$$\pi\Lambda(\tau_i' \cap N_{t_2}p, t_2) \cap \pi(\tau_i^r - N_{t_1}p) = 0.$$

The last two relations give

$$\pi\Lambda(\tau_i' \cap N_{t_2}p, t_2) \cap \pi(\tau_i^r) = 0,$$

and hence (with $t_3 \leqq t_2$)

$$\pi\Lambda(\tau_i' \cap N_{t_3}\partial\tau_i^r, t_3) \cap \pi(\tau_i^r) = 0,$$

proving (c_6). Next, since $(\tau_i' - N_{t_3}\partial\tau_i^r) \cap K^{r-1} = 0$, ($b_2^*$) gives, for some $t_4 \leqq t_3$,

$$\pi\Lambda(\tau_i' - N_{t_3}\partial\tau_i^r, t_4) \cap \pi(\partial\tau_i^r) = 0.$$

Combining this with the last relation gives

$$\pi\Lambda(\tau_i', t_4) \cap \pi(\partial\tau_i^r) = 0,$$

proving (c_5).

REMARK. With the help of (b_5) we could prove (recalling that K is proper)

$$\pi\Lambda_{v_k}(\tau_i', t_i) \cap \pi K^{r-1} = 0.$$

18. The numbers $\xi_{i,v}^{\pm}$ and $\zeta_{ij}^{\pm}$

We give here, and with the $\mu_{ij,v}^{\pm}$ of §19, the promised generalization of the looping coefficient of §14. Let K^r and each v_i be in general position in E^{2r+1} with reference to the y_{2r+1}-direction, as in §17. For any (finite) singular chain A in E, let ρ^+A and ρ^-A be the (infinite) singular chains formed by deforming A to infinity in the y_{2r+1}-direction and the negative y_{2r+1}-direction respectively, oriented so that

$$\partial\rho^{\pm}A = -A - \rho^{\pm}\partial A. \tag{18.1}$$

We shall show that the following definitions are permissible.

DEFINITIONS. Choose the t_i so that (c_1) through (c_6) hold. Let $\phi_{v,t}A$ denote the singular chain A, pushed a distance $t \mid v \mid$ in the direction of v, and oriented like A; see (17.1). Set

$$\xi_{i,v_k}^{\pm} = KI(\phi_{v_k,t_i}\tau_i^r, \rho^{\pm}\tau_i^r) \quad \text{(if } v_k \text{ is defined in } \tau_i^r\text{)}, \tag{18.2}$$

$$\zeta_{ij}^{\pm} = KI(\tau_i^*, \rho^{\pm}\tau_j^*) \qquad (i \neq j), \tag{18.3}$$

where τ_i^* is any cell lying in τ_i^r and slightly smaller than τ_i^r, oriented like τ_i^r. Note that $\xi_{i,v_k}^{\pm}$ does not depend on the orientation of τ_i^r.

We shall prove commutation properties of these:

$$\xi_{i,v_k}^{+} = (-1)^{r+1}\xi_{i,-v_k}^{-}, \tag{18.4}$$

$$\zeta_{ij}^{+} = (-1)^{r+1}\zeta_{ji}^{-}. \tag{18.5}$$

In the proofs of these and other relations, the following properties of Kronecker indices are useful:

(d_1) $KI(A, \rho^{\pm}B)$ is defined (for finite A and B) whenever $\dim A + \dim B = 2r$, and

$$A \cap B = 0, \qquad \pi A \cap \pi\partial B = 0, \qquad \pi\partial A \cap \pi B = 0.$$

We must show that

$$\partial A \cap \rho^{\pm}B = 0, \qquad A \cap \partial\rho^{\pm}B \subset (A \cap B) \cup (A \cap \rho^{\pm}\partial B) = 0;$$

these follow from the above relations.

(d_2) If A_λ and B are singular chains, A_λ being continuous in λ ($\lambda_0 \leqq \lambda \leqq \lambda_1$), and each $KI(A_\lambda, B)$ is defined, then $KI(A_{\lambda_0}, B) = KI(A_{\lambda_1}, B)$.

This follows either from continuity, or by making use of chains formed by deforming A_{λ_0} and ∂A_{λ_0}, with standard properties of the index.

(d_3). If $KI(A, B)$ is defined, and C_λ ($\lambda_0 \leqq \lambda \leqq \lambda_1$) satisfies $C_{\lambda_0} = \partial A$, $C_\lambda \cap B = 0$, then this defines a deformation A_λ of A such that $\partial A_\lambda = C_\lambda$ and $KI(A_\lambda, B)$ is defined.

This is clear if we let A_λ equal A plus the "path" of $C_{\lambda'}$, $\lambda_0 \leqq \lambda' \leqq \lambda$.

To prove (18.4), and show incidentally that (18.2) is permissible, choose l so that $-v_k = v_l$. By (d_1), we see that each of

$$KI(\phi_{v_k, t_i}\tau_i^r, \rho^{\pm}\phi_{-v_k, t}\tau_i^r), \qquad KI(\phi_{v_k, t}\tau_i^r, \rho^{\pm}\phi_{-v_k, t_i}\tau_i^r)$$

is defined for $0 \leqq t \leqq t_i$; for the three relations in (d_1) follow from (c_2), (c_3) and (c_5), using both v_k and v_l. Consequently, by (d_2),

$$\xi^+_{i,v_k} = KI(\tau_i^r, \rho^+\phi_{-v_k, t_i}\tau_i^r) = KI(\rho^+\phi_{-v_k, t_i}\tau_i^r, \tau_i^r).$$

In the above proof, if $A = \phi_{-v_k, t_i}\tau_i^r$, we used $\pi A \cap \pi\partial\tau_i^r = \pi\partial A \cap \pi\tau_i^r = 0$. These give

$$KI(\rho^+A, \rho^-\partial\tau_i^r) = KI(\rho^+\partial A, \rho^-\tau_i^r) = 0,$$

and therefore, since $\dim \rho^+A = r + 1$,

$$\begin{aligned}\xi^+_{i,v_k} &= -KI(\rho^+A, -\tau_i^r - \rho^-\partial\tau_i^r) = -KI(\rho^+A, \partial\rho^-\tau_i^r)\\ &= (-1)^r KI(\partial\rho^+A, \rho^-\tau_i^r) = (-1)^{r+1}KI(A + \rho^+\partial A, \rho^-\tau_i^r)\\ &= (-1)^{r+1}KI(\phi_{-v_k, t_i}\tau_i^r, \rho^-\tau_i^r) = (-1)^{r+1}\xi^-_{i,-v_k}.\end{aligned}$$

Next we discuss (18.3) and (18.5). Because of (b_6^*), ζ^+_{ij} is defined and independent of the choice of any τ_k^*, so long as $\tau_k^r - \tau_k^* \subset V^*$. Since

$$\pi\tau_i^* \cap \pi\partial\tau_j^* = \pi\partial\tau_i^* \cap \pi\tau_j^* = 0,$$

we find, as in the proof of (18.4),

$$\begin{aligned}\zeta^+_{ij} &= KI(\rho^+\tau_j^*, \tau_i^*) = -KI(\rho^+\tau_j^*, -\tau_i^* - \rho^-\partial\tau_i^*)\\ &= -KI(\rho^+\tau_j^*, \partial\rho^-\tau_i^*) = (-1)^r KI(\partial\rho^+\tau_j^*, \rho^-\tau_i^*)\\ &= (-1)^{r+1}KI(\tau_j^* + \rho^+\partial\tau_j^*, \rho^-\tau_i^*) = (-1)^{r+1}\zeta^-_{ji}.\end{aligned}$$

We shall prove still a lemma regarding the $\xi^\pm_{i,v_k}$. Since $\bar\tau_i^r = \theta(\bar\sigma_0^r)$, $\bar\sigma_0^r$ convex in E^r, the parts near $\partial\sigma_0^r$ of radii from an inner point of σ_0^r map under θ into arcs which we may suppose fill out $\tau_i' \cap N_{t_i}\partial\tau_i^r$. With these arcs, we may define a deformation g_λ such that

(e_1) $g_\lambda(p)$ carries p along the arc in $\tau_i' \cap N_{t_i}\partial\tau_i^r$ on which it lies into $\partial\tau_i^r$.

LEMMA 19. *Let A be any singular chain such that (writing N for N_{t_i} and $\Lambda_v(R)$ for $\Lambda_v(R, t_i)$)*

(18.6) $$A \subset \Lambda_{v_k}(\tau_i' \cap N\tau_i^r), \qquad \partial A \subset \Lambda_{v_k}(\tau_i' \cap N\partial\tau_i^r).$$

Then there is a number a and a chain B such that

(18.7) $$KI(A, \rho^{\pm}\tau_i^r) = a\xi^{\pm}_{i,v_k},$$

(18.8) $$\partial A - a\partial\tau_i^r = \partial B, \qquad B \subset \Lambda_{v_k}(\tau_i' \cap N\partial\tau_i^r) \cup (\tau_i' \cap N\partial\tau_i^r).$$

The last relation determines a uniquely, even if we assume merely $B \subset W_{v_k}(\tau_i' \cap N\partial\tau_i^r, t_i)$.

First, by (c_4), (c_5), (c_6) and (d_1), $KI(A, \rho^{\pm}\tau_i^r)$ is defined. We shall define a deformation h_λ of A $(0 \leqq \lambda \leqq 3)$ such that

(18.9) $$h_3 A = a\phi_{v_k,t_i}\tau_i^r.$$

Also $KI(h_\lambda A, \rho^{\pm}\tau_i^r)$ is defined for each λ; (18.7) follows from this, together with (18.2) and (d_2).

For any $q \in \Phi^*_{v_k}(\tau_i' \cap N\tau_i^r)$ and $q' \in T_0(q)$, set

$$h_\lambda(q') = (1 - \lambda)q' + \lambda q \qquad (0 \leqq \lambda \leqq 1);$$

this is defined in $\Lambda_{v_k}(\tau_i' \cap N\tau_i^r)$, by (c_1), and $h_1(q') = P_{v_k}(q)$, by (17.2). From the definition of Λ_{v_k} we see that $h_\lambda A$ obeys (18.6) so far, simply because A does. Moreover, $h_1 A$ is in the deleted fin $\Phi^*_{v_k}(\tau_i' \cap N\tau_i^r)$, and $h_1\partial A \subset \Phi^*_{v_k}(\tau_i' \cap N\partial\tau_i^r)$.

Next, for each $q = \phi_{v_k,t}(p)$ $(p \in \tau_i')$ in the deleted fin, set

$$h_{1+\lambda}(q) = (1 - \lambda)q + \lambda\phi_{v_k,t_i}(p) \qquad (0 \leqq \lambda \leqq 1);$$

this is a deformation of $\Phi^*_{v_k}(\tau_i')$ in itself; applying it to $h_1 A$ defines $h_\lambda A$ $(0 \leqq \lambda \leqq 2)$ so that (18.6) continues to hold; now

$$h_2 A \subset \phi_{v_k,t_i}(\tau_i' \cap N\tau_i^r), \qquad h_2\partial A \subset \phi_{v_k,t_i}(\tau_i' \cap N\partial\tau_i^r).$$

Next, applying ϕ_{v_k,t_i} to the deformation g_λ defines h_λ, $2 \leqq \lambda \leqq 3$; by (e_1), it keeps $\phi_{v_k,t_i}(\tau_i' \cap N\partial\tau_i^r)$ in itself. This defines a deformation of $\partial h_2 A$, and hence of $h_2 A$, so that (18.6) continues to hold; see (d_3). Now

$$h_3 A \subset \phi_{v_k,t_i}(\tau_i^r), \qquad h_3\partial A = \partial h_3 A \subset \phi_{v_k,t_i}(\partial\tau_i^r).$$

Since ϕ_{v_k,t_i} is one-one in τ_i^r, the only $(r-1)$-cycles in $\phi_{v_k,t_i}(\partial\tau_i^r)$ are multiples of the cycle $\phi_{v_k,t_i}\partial\tau_i^r$; this proves (18.9) and hence (18.7).

The deformation h_λ of ∂A $(0 \leqq \lambda \leqq 3)$ defines a chain B_1 such that

$$\partial B_1 = \partial A - a\phi_{v_k,t_i}\partial\tau_i^r;$$

since $\Phi_{v_k}(\partial\tau_i^r)$, properly oriented, is a chain B_2 bounded by $\phi_{v_k,t_i}\partial\tau_i^r - \partial\tau_i^r$, $B = B_1 + aB_2$ satisfies (18.8).

To prove the uniqueness of a in (18.8), suppose it held with a' and B' also; then

$$(a' - a)\partial\tau_i^r = \partial C, \qquad C = B - B' \subset W_{v_k}(\tau_i' \cap N\partial\tau_i^r, t_i).$$

If we contract the fin $\Phi_{v_k}(\tau_i')$ onto τ_i', we may carry C into $C' \subset \tau_i'$, and the above holds with C' in place of C. Since $N_{t_i}\partial\tau_i^r$ does not cover all of τ_i^r, this implies $a' - a = 0$.

19. Application to complexes K^n mapped into E^{2n-1}

In this section we suppose that f, of class C^3, maps K^n in regular position in E^{2n-1} with reference to the y_{2n-1}-direction. We may suppose K imbedded in E^{2n+1}. Extend each cell $\bar\sigma_i^s$ to a cell $\sigma_i'^s$ as in Lemma 16; we may extend f over these in turn so that the properties in Lemma 16 and the remark following it hold. Set $\tau_i'^s = f(\sigma_i'^s)$. Taking these cells small enough, we may suppose the properties (b_1), (b_2), (b_3) still hold.

We shall determine open sets U_i and U' in K (which now consists of the $\sigma_i'^s$) such that:

(f_1) $\bar\sigma_i^{n-1} \subset U_i$, $K^{n-2} \subset U'$.

(f_2) $U_i \cap U_j \subset U'$ if $i \neq j$.

(f_3) f is an imbedding in $U = \sum U_i$.

(f_4) If $p \in U$, $p' \in U'$, $p \neq p'$, then $\pi f(p) \neq \pi f(p')$.

We shall restrict the U_i and U' further later. That f is locally an imbedding at points of K^{n-1} is an immediate consequence of (b_1) and (b_3), using $s = n$, $\nu - s = n - 1$. The proof of (f_3) and (f_4) is now the same as the proof of (b_6^*). It is easy to choose sets U_i satisfying (f_1) and (f_2) (and thus redefining U) if we make use of the sets $\sigma_j^{n-1} - U'$.

Set $r = n - 1$. For each pair of incident cells σ_i^r and σ_k^n, let $\sigma_{ik}'^n$ be the part of $\sigma_k'^n$ on the side of $\sigma_i'^r$ which includes σ_k^n, and let $u_{ik}(p)$ ($p \in \sigma_i'^r$) be the unit vector tangent to $\sigma_{ik}'^n$ and normal to $\sigma_i'^r$ at $p \in \sigma_i'^r$. Set $v_{ik}(p) = \nabla f(u_{ik}(p), p)$. The following property shows that we may use the results of §§17 and 18.

(f_5) f and each v_{ik}, also f and each $-v_{ik}$, are in general position with reference to the y_{2n-1}-direction, (b_5) being omitted. (Here, f is considered in K^{n-1} only.)

We need merely prove (b_4) for f and πf; this follows at once from the properties of general position; compare (f_3) and (f_4).

Set $V_i = f(U_i)$, $V = f(U)$, $V' = f(U')$.

Next, by further restricting U' and then the U_i, we may obtain:

(f_6) $\tau_{ik}'^n \cap V_i \subset \Lambda_{v_{ik}}(\tau_i'^r \cap N_{t_i}\tau_i^r, t_i)$,

(f_7) $\tau_{ik}'^n \cap V' \subset \Lambda_{v_{ik}}(\tau_i'^r \cap N_{t_i}\partial\tau_i^r, t_i)$.

These are obvious consequences of the definitions of $\tau_{ik}'^n$, v_{ik} and Λ_v.

Now let K' be a subdivision of K so fine that the following holds:

(f_8) Any cell of K' with a vertex in $\bar\sigma_i^r$ lies in U_i; any cell with a vertex in K^{r-1} lies in U'.

Each oriented cell σ of K becomes now a chain $Sd\sigma$ of K'. Let $[\sigma_i^{s-1}\colon \sigma_k^s]$ denote incidence numbers. Define the following chains of K', and use $\tau = f(\sigma)$ as before:

σ_i^{*s} = sum of s-cells of K' in σ_i^s with no vertex in $\partial\sigma_i^s$, each oriented like σ_i^s.

σ_{ik}^{*} = sum of n-cells of σ_{ik}^n which have a vertex in σ_i^r but none in $\partial\sigma_i^r$, each oriented like $[\sigma_i^r\colon \sigma_k^n]\sigma_k^n$.

σ_{ik}^{**} = that part of $-\partial\sigma_{ik}^{*}$ whose cells have no vertex in $\bar\sigma_i^r$.

We prove the following relations:

$$\partial\tau_{ik}^{*} = \tau_i^{*r} - \tau_{ik}^{**} + A_{ik}, \qquad A_{ik} \subset V'. \tag{19.1}$$

$$\partial\tau_k^{*n} = \sum_i [\sigma_i^r : \sigma_k^n]\tau_{ik}^{**} + B_k, \qquad B_k \subset V'. \tag{19.2}$$

First, supposing σ_i^r a face of σ_k^n, note that each cell of the chain

$$Sd\tau_k^n - (\tau_k^{*n} + [\sigma_i^r : \sigma_k^n]\tau_{ik}^{*})$$

has a vertex in $\partial\tau_k^n - \tau_i^r$; hence the chain lies in V_i', where $U_i' = \sum_{j\neq i} U_j$, $V_i' = f(U_i')$. Also, clearly

$$\partial Sd\tau_k^n = Sd\partial\tau_k^n = [\sigma_i^r : \sigma_k^n]\tau_i^{*} + C_{ik}, \qquad C_{ik} \subset V_i'.$$

It follows that

$$\partial(\tau_k^{*n} + [\sigma_i^r : \sigma_k^n]\tau_{ik}^{*}) = [\sigma_i^r : \sigma_k^n]\tau_i^{*r} + D_{ik}, \qquad D_{ik} \subset V_i'.$$

This last relation, with (19.1), gives

$$\partial\tau_k^{*n} = [\sigma_i^r : \sigma_k^n]\tau_{ik}^{**} + D_{ik} \pm A_{ik}.$$

Now neither τ_k^{*n} nor τ_{ik}^{**} have any cells with vertices in $\bar{\tau}_i^r$. Hence neither has $D_{ik} \pm A_{ik}$, and it follows that each cell of A_{ik} is a cell of D_{ik}. Therefore $A_{ik} \subset V_i'$. Since τ_{ik}^{*}, τ_i^{*r} and τ_{ik}^{**} are in V_i, each point of A_{ik} is in some $V_i \cap V_j = f(U_i \cap U_j)$ (see (f_3)) with $j \neq i$, and therefore in V', by (f_2). This proves (19.1).

Next, since

$$\sum_{j\neq i} [\sigma_j^r : \sigma_k^n]\tau_{jk}^{**}, \quad D_{ik}, \quad A_{ik} \text{ are in } V_i',$$

the last relation for $\partial\tau_k^{*n}$ gives

$$B_k = \partial\tau_k^{*n} - \sum_l [\sigma_l^r : \sigma_k^n]\tau_{lk}^{**} \subset V_i'.$$

Since B_k is independent of i, this holds for each i. Now if B_k had a cell with a point p in just one V_{i_0} (clearly $B_k \subset \sum V_j$), using i_0 in the last relation would give a contradiction. Hence each p is in at least two, and hence in V'. This proves (19.2).

With the help of the new chains, we shall find new expressions for the numbers in §18:

$$\xi_{i,v_{ik}}^{\pm} = KI(\tau_{ik}^{**}, \rho^{\pm}\tau_i^r) \qquad ([\sigma_i^r : \sigma_k^n] \neq 0), \tag{19.3}$$

$$\zeta_{ij}^{\pm} = KI(\tau_{ik}^{**}, \rho^{\pm}\tau_j^r) \qquad ([\sigma_i^r : \sigma_k^n] \neq 0, i \neq j). \tag{19.4}$$

First, since each cell of σ_{ik}^{**} is a face of a cell of σ_{ik}^{*}, which has a vertex in σ_i^r, (f_8) gives:

$$\tau_{ik}^{**} \subset \tau_{ik}'^{n} \cap V_i. \tag{19.5}$$

Taking the boundary of (19.1) and using (f_8) gives

$$\partial\tau_{ik}^{**} = \partial\tau_i^{*r} + \partial A_{ik} \subset \tau_{ik}'^{n} \cap V'. \tag{19.6}$$

These relations, with (f_6) and (f_7), show that we may apply Lemma 19, which gives

$$KI(\tau_{ik}^{**}, \rho^{\pm}\tau_i^r) = a\xi_{i,v_{ik}}^{\pm}.$$

We must show that $a = 1$. By (19.6),

$$\partial\tau_{ik}^{**} - \partial\tau_i^r = \partial[A_{ik} - (\tau_i^r - \tau_i^{*r})].$$

By (f_8) and (19.1), $A_{ik} - (\tau_i^r - \tau_i^{*r}) \subset V'$. Since this chain is also in $\tau_i'^r \cup \tau_{ik}'^n$, applying ($f_7$) gives (18.8) with $a = 1$. The uniqueness of a in (18.8) completes the proof of (19.3).

Next, taking $i \neq j$, since

$$\tau_j^r - \tau_j^{*r} \subset V', \qquad \tau_{ik}^{**} \subset V, \qquad (\tau_j^r - \tau_j^{*r}) \cap \tau_{ik}^{**} = 0,$$

(f_4) gives $\pi(\tau_j^r - \tau_j^{*r}) \cap \pi(\tau_{ik}^{**}) = 0$. Hence

$$KI(\tau_{ik}^{**}, \rho^{\pm}\tau_j^r - \rho^{\pm}\tau_j^{*r}) = 0.$$

Similarly, using (19.1), we find the two relations

$$KI(A_{ik}, \rho^{\pm}\tau_j^{*r}) = KI(\tau_{ik}^{*}, \rho^{\pm}\partial\tau_j^{*r}) = 0.$$

By (f_3), $KI(\tau_{ik}^{*}, \tau_j^{*r}) = 0$. This, with the last relations, gives

$$0 = KI(\tau_{ik}^{*}, \partial\rho^{\pm}\tau_j^{*r}) = \pm KI(\tau_i^{*r} - \tau_{ik}^{**}, \rho^{\pm}\tau_j^{*r}).$$

Hence

$$KI(\tau_{ik}^{**}, \rho^{\pm}\tau_j^r) = KI(\tau_{ik}^{**}, \rho^{\pm}\tau_j^{*r}) = KI(\tau_i^{*r}, \rho^{\pm}\tau_j^{*r}),$$

which gives (19.4).

We now give an extension of the results in §14, in particular, of Lemma 13.

DEFINITION. Let K^r and v be in general position in E^{2r+1}, with reference to the y_{2r+1}-direction. With the notations of §18, and a sufficiently small positive continuous $\eta(p)$ in K, set

$$\mu_{ij,v}^{\pm} = KI(\phi_{v,\eta}\tau_i^r, \rho^{\pm}\tau_j^r). \tag{19.7}$$

Save possibly for sign, this is a direct generalization of (14.2). The commutation rule is:

$$\mu_{ij,v}^{+} = (-1)^{r+1}\mu_{ji,-v}^{-}. \tag{19.8}$$

The proof is easily carried out, with the help particularly of (b_5), as follows:

$$\begin{aligned}\mu_{ij,v}^{+} &= KI(\rho^{+}\tau_j^r, \phi_{v,\eta}\tau_i^r) = KI(\rho^{+}\phi_{-v,\eta}\tau_j^r, \tau_i^r)\\ &= -KI(\rho^{+}\phi_{-v,\eta}\tau_j^r, \partial\rho^{-}\tau_i^r) = (-1)^r KI(\partial\rho^{+}\phi_{-v,\eta}\tau_j^r, \rho^{-}\tau_i^r)\\ &= (-1)^{r+1}KI(\phi_{-v,\eta}\tau_j^r, \rho^{-}\tau_i^r) = (-1)^{r+1}\mu_{ji,-v}^{-}.\end{aligned}$$

DEFINITION. With the above notations, set

$$\mathfrak{L}^{\pm}(A, B, v) = KI(\phi_{v,\eta}A, \rho^{\pm}B). \tag{19.9}$$

The commutation rule, generalizing (14.3), is

$$\mathfrak{L}^{+}(A, B, v) = (-1)^{r+1}\mathfrak{L}^{-}(B, A, -v). \tag{19.10}$$

To prove this, suppose $A = \sum a_i\tau_i^r$, $B = \sum b_j\tau_j^r$. Then, writing ϕ for $\phi_{v,\eta}$,

$$\begin{aligned}\mathfrak{L}^{+}(A, B, v) &= \sum_{i,j} a_i b_j KI(\phi\tau_i^r, \rho^{+}\tau_j^r)\\ &= \sum_i a_i b_i \xi^{+}_{i,v} + \sum_{i<j}(a_i b_j \mu^{+}_{ij,v} + a_j b_i \mu^{+}_{ji,v})\\ &= (-1)^{r+1}[\sum_i b_i a_i \xi^{-}_{i,-v} + \sum_{i<j}(b_i a_j \mu^{-}_{ij,-v} + b_j a_i \mu^{-}_{ji,-v})]\end{aligned}$$

which, by the same proof, equals $(-1)^{r+1}\mathfrak{L}^{-}(B, A, -v)$.

20. Application to partial manifolds M^n mapped into E^{2n-1}

The following theorem contains essentially Theorem 2 if n is odd. Let $\mathfrak{L}'_f(M)$ denote the algebraic number of singular points if n is odd.

THEOREM 9. *Let M^n be a partial manifold, let K be a subdivision of M (with the properties of §15), and let f be a semi-regular mapping of M into E^{2n-1} so that f maps K in general position. For each σ_i^{n-1} in ∂M, let $v_i(p)$ point into M at $f(p)$, $p \in \bar{\sigma}_i^{n-1}$. Then letting $\sum'$ denote the sum taken over just those cells in ∂M, we have:*

(a) *If n is even,*

$$\sum{}' \xi^{+}_{i,v_i} = \sum{}' \xi^{-}_{i,v_i} = \sum{}' \xi^{+}_{i,-v_i}. \tag{20.1}$$

(b) *If n is odd,*

$$2\mathfrak{L}'_f(M) = \sum{}' (\xi^{+}_{i,v_i} + \xi^{-}_{i,v_i}) = \sum{}' (\xi^{+}_{i,v_i} - \xi^{+}_{i,-v_i}). \tag{20.2}$$

We consider first the case that M^n is a single cell σ_k^n. Using the previous notations, setting $\partial_{ik} = [\sigma_i^{n-1}:\sigma_k^n]$ and noting that $\partial_{ik}^2 = 0$ or 1, we find, with the help of (19.2), (19.3) and (19.4),

$$\begin{aligned}\mathfrak{L}_f(\sigma_k^n) &= KI(\tau_k^{*n}, \partial\tau_k^n) = -KI(\tau_k^{*n}, \partial\rho^{\pm}\partial\tau_k^n)\\ &= (-1)^{n-1}KI(\partial\tau_k^{*n}, \rho^{\pm}\partial\tau_k^n) = (-1)^{n-1}\sum_{i,j}\partial_{ik}\partial_{jk}KI(\tau_{ik}^{**}, \rho^{\pm}\tau_j^r)\\ &= (-1)^{n-1}[\sum_i \xi^{\pm}_{i,v_{ik}} + \sum_{i<j}\partial_{ik}\partial_{jk}(\zeta^{\pm}_{ij} + \zeta^{\pm}_{ji})].\end{aligned}$$

Using first $+$ and then $-$, we find, with the help of (18.5),

$$(-1)^{n-1}\mathfrak{L}_f(\sigma_k^n) = \sum_i \xi^{+}_{i,v_{ik}} + \sum_{i<j}\partial_{ik}\partial_{jk}(\zeta^{+}_{ij} + \zeta^{+}_{ji}),$$

$$\mathfrak{L}_f(\sigma_k^n) = (-1)^{n-1}\sum_i \xi^{-}_{i,v_{ik}} - \sum_{i<j}\partial_{ik}\partial_{jk}(\zeta^{+}_{ji} + \zeta^{+}_{ij}).$$

Adding these and using Theorem 2, gives

$$[1 + (-1)^{n-1}]\mathfrak{L}'_f(\sigma_k^n) = \sum_i [\xi^+_{i,v_{ik}} + (-1)^{n-1}\xi^-_{i,v_{ik}}], \tag{20.3}$$

from which the theorem for this case follows, with the help of (18.4).

By (20.3) and (18.4), we have, in the general case,

$$[1 + (-1)^{n-1}] \sum_k \mathfrak{L}'_f(\sigma_k^n) = \sum_i \sum_k (\xi^+_{i,v_{ik}} - \xi^+_{i,-v_{ik}}),$$

where for each i we sum over all k such that σ_i^{n-1} is a face of σ_k^n. If σ_i^{n-1} is interior to M, then σ_i^{n-1} is a face of two cells σ_k^n and σ_l^n, and $v_{il} = -v_{ik}$; hence

$$(\xi^+_{i,v_{ik}} - \xi^+_{i,-v_{ik}}) + (\xi^+_{i,v_{il}} - \xi^+_{i,-v_{il}}) = 0.$$

Thus all these values of i drop out. For each i with $\sigma_i^{n-1} \subset \partial M$, there is just one k, and $v_i = v_{ik}$. Hence the theorem follows.

REMARK. Clearly both sides of (20.2) are independent of the subdivision K of M employed. All the terms are independent of the chosen orientations of cells.

DEFINITION. Let M be a partial n-manifold, n odd, with the property that there exists a continuous vector field $u(p)$ defined in ∂M, $u(p)$ being independent of each closed cell of ∂M containing p, and pointing into M. Let K be a subdivision of M and let f be a semi-regular mapping of M into E^{2n-1} such that f maps K in general position with reference to the y_{2n-1}-direction. Set $v(p) = \nabla f(u, p)$. Set

$$A^{n-1} = \sum' \sigma_i^{n-1}, \text{ summed over the cells in } \partial M, \tag{20.4}$$

these cells being oriented arbitrarily. Let e be the unit vector in the y_{2n-1}-direction. Then for $\alpha(p) > 0$ sufficiently small in ∂M, and $\eta(p) > 0$ sufficiently much smaller than $\alpha(p)$, define (using the definition of ψ in §14 and recalling that n is odd)

$$\begin{aligned} \mathfrak{L}_f(M) &= \tfrac{1}{2}[KI(\phi_{v,\eta}fA, \psi_{e,0,\alpha}fA) + KI(\phi_{v,\eta}fA, \psi_{-e,0,\alpha}fA)] \\ &= \tfrac{1}{2}[KI(\phi_{v,\eta}fA, \psi_{e,0,\alpha}fA) - KI(\phi_{-v,\eta}fA, \psi_{e,0,\alpha}fA)]. \end{aligned} \tag{20.5}$$

REMARK. If the hypothesis of general position in Theorem 9 does not hold, we may apply Lemma 17 to make it hold; that $\mathfrak{L}_f(M)$ is independent of the rotation chosen follows from the proof below, in which general position is assumed.

COMPLETION OF THE PROOF OF THEOREM 2. We must prove $\mathfrak{L}'_f(M) = \mathfrak{L}_f(M)$. Because of (20.2), it is only necessary to show that (for n odd)

$$KI(\phi_{v,\eta}fA, \psi_{e,0,\alpha}fA) = \sum' \xi^+_{i,v_i}. \tag{20.6}$$

For each σ_i^{n-1} in ∂M, we may deform $v_i(p)$ into $v(p)$, keeping it tangent to M and independent of $\bar{\sigma}_i^{n-1}$; hence ξ^+_{i,v_i} may be replaced by $\xi^+_{i,v}$. Since both sides of (20.6) are independent of the subdivision employed, and πf is regular in K^{n-1} and hence in ∂M, we may suppose that K is so fine that for each pair σ_i^{n-1},

σ_j^{n-1} of cells of ∂M with a common vertex, πf is an imbedding in $\bar{\sigma}_i^{n-1} \cup \bar{\sigma}_j^{n-1}$. Since also πf is regular in M at points of K^{n-2}, it follows at once that for small enough $\alpha(p)$ and $\eta(p)$,

$$KI(\phi_{v,\eta}\tau_i^r , \psi_{e,0,\alpha}\tau_j^r) = 0 \qquad (i \neq j).$$

Exactly as in corresponding proofs in §19, we see that

$$KI(\phi_{v,\eta}\tau_i^r , \psi_{e,0,\alpha}\tau_i^r) = KI(\phi_{v,\eta}\tau_i^r , \rho^+\tau_i^r) = \xi_{i,v}^+ .$$

We find therefore

$$KI(\phi_{v,\eta}fA, \psi_{e,0,\alpha}fA) = \sum_{i,j}' KI(\phi_{v,\eta}\tau_i^r , \psi_{e,0,\alpha}\tau_j^r)$$

$$= \sum_i' KI(\phi_{v,\eta}\tau_i^r , \psi_{e,0,\alpha}\tau_i^r) = \sum_i' \xi_{i,v}^+ ,$$

taken over the cells of ∂M, which completes the proof.

21. The necessity for the type of formula in (20.5) (n odd)

One might expect that a single term on the right in (20.5) would suffice without the factor $\frac{1}{2}$. This is the case if the cells σ_i^r of ∂M can be so oriented that $\sum'\sigma_i^r$ is a cycle, but not in the general case, as we shall show.

To compare the two terms, note that

$$\psi_{-e,0,\alpha} = -\psi_{-e,\alpha,0} = -\psi_{e,-\alpha,0};$$

hence, subtracting one of the terms from the other,

$$\begin{aligned} \Delta &= KI(\phi_{v,\eta}fA, \psi_{e,0,\alpha}fA - \psi_{-e,0,\alpha}fA) \\ &= KI(\phi_{v,\eta}fA, \psi_{e,-\alpha,\alpha}fA). \end{aligned} \tag{21.1}$$

Let e' be the unit vector in the y_{2n-2}-direction. By deforming $\phi_{v,\eta}\, C$ into $\phi_{e',\eta}C$, we define θC for all $C \subset \partial M$, with the property

$$\partial\theta C = \phi_{e',\eta}C - \phi_{v,\eta}C - \theta\partial C;$$

we may take this in general position together with the chains ψ considered, with reference to the y_{2n-1}-direction. From the obvious relations (for small $\eta(p)$)

$$\phi_{e',\eta}fA \cap \psi_{e,-\alpha,\alpha}fA = 0,$$

$$\theta fA \cap (\phi_{e,\alpha}fA \cup \phi_{e,-\alpha}fA) = 0,$$

we obtain

$$\begin{aligned} \Delta &= KI(\phi_{v,\eta}fA - \phi_{e',\eta}fA, \psi_{e,-\alpha,\alpha}fA) \\ &= -KI(\partial\theta fA + \theta\partial fA, \psi_{e,-\alpha,\alpha}fA) \\ &= -KI(\theta\partial fA, \psi_{e,-\alpha,\alpha}fA) - (-1)^n KI(\theta fA, \partial\psi_{e,-\alpha,\alpha}fA), \\ \Delta &= -KI(\theta f\partial A, \psi_{e,-\alpha,\alpha}fA) + (-1)^n KI(\theta fA, \psi_{e,-\alpha,\alpha}f\partial A). \end{aligned} \tag{21.2}$$

As a corollary, $\Delta = 0$ if the cells σ_i^r of ∂M can be so oriented that their sum is a cycle.

REMARK. The last fact follows also on applying Lemma 14 to $\mathfrak{L}(fA, fA, v, \pm e)$.

EXAMPLE. We shall show that Δ may be $\neq 0$. Take a cylinder plus interior:

$$x^2 + y^2 \leqq 1, \quad -1 \leqq z \leqq 1;$$

identify the two ends, after reflecting one across a diameter: set

$$(x, y, -1) = (x, -y, 1).$$

A "Klein bottle B^2 plus interior" is formed; this is a partial manifold M^3, with $\partial M = B^2$. Let K be a subdivision of M^3. We may imbed B^2 in E^4 in such a fashion that for any continuous vector field $v(p)$ in B^2, independent of B^2 at points of $K^1 \cap B^2$,

$$KI(\phi_{v,\eta} B_0^2, B_0^2) = -4 \text{ in } E^4,$$

B_0^2 being $\sum' \sigma_i^2$, σ_i^2 in B^2, and η being small; see [3], p. 107, Fig. 4, and the top of p. 108. Now take $E^4 \subset E^5$. Then clearly, from the above,

$$\Delta = KI(\phi_{v,\eta} B_0^2, \psi_{e,-\alpha,\alpha} B_0^2) = -4 \text{ in } E^5.$$

If, for instance, we take $v = (e + e')/2^{1/2}$, then (20.5) reads

$$\mathfrak{L}_f(M^3) = \tfrac{1}{2}[-4 + 0] = -2.$$

Mapping M^3 into E^5 so that v points into $f(M)$ gives the result stated.

APPENDIX

22. The self-intersections under a completely semi-regular mapping

Let A be the set of points p of M such that for some $q \neq p$, $f(p) = f(q)$. We shall discuss A and $f(A)$.

Let $p_1, p_2, \cdots$ be the singular points. By Lemma 2 and the definition of completely semi-regularity, there is a neighborhood U_i^* of p_i such that $A \cap U_i^*$ is given by $x_2 = \cdots = x_n = 0$; thus this is an open arc, mapped by f doubly into a half-open arc. We may suppose $\overline{U}_i^* \cap \overline{U}_j^* = 0$ for $i \neq j$.

Choose U_i^{**}, $p_i \in U_i^{**}$, $\overline{U}_i^{**} \subset U_i^*$, and set $M' = M - \sum \overline{U}_i^{**}$. Take p, q in M', $p \neq q$, $f(p) = f(q)$. If $p, q \in M' - \partial M'$, there are neighborhoods U and V of p and q in M' such that $f(U) \cap f(V)$ is a smooth open arc in E, the image of smooth open arcs in M'; if one of p, q is in $\partial M'$, these are half-open arcs. We may cover M' by neighborhoods U_i such that each $f(U_i) \cap f(U_j)$ $(U_i \cap U_j = 0)$ is void or a smooth open or half-open arc. These arcs in M' together with those in $\sum \overline{U}_i^{**}$ fit together to form simple closed curves or arcs; in each direction on each arc, we end either at a singular point or at a point whose image is also the image of a point of ∂M, or we reach no limit point of M (this can occur only if M is open). Moreover, it is not hard to see that there is a grouping of these curves into pairs (the two in a pair need not be distinct), in one of the following ways:

(a) There may be two distinct closed curves, with one closed curve as image.

(b) A single closed curve may be mapped doubly over a closed curve.

(c) If the closed curve contains any singular points, it contains two, and is mapped into an arc.

(d) An arc may have one singular point interior to it; then (d_1) it is open at both ends, or (d_2) one end stops at ∂M and the other stops interior to M.

(e) A pair of arcs, each containing no singular points, may map into an arc. Then (e_1) both are open, or (e_2) each is half open, one ends in ∂M, and the other ends interior to M, or (e_3) one has both ends in ∂M and the other has both ends interior to M, or (e_4) each has one end in ∂M and one end interior to M.

If $n = 2$, we have the same subdivision into cases, but the curves may cut through each other. The case $n = 1$ is trivial. We could further subdivide the cases by taking into account orientation properties.

REMARK. If f is not proper, the set A need not be closed; for example, we may map a strip M^2 around and around in the interior of a torus so that A is dense in M^2. Still A is expressible as a union of curves.

23. On the covering of an open partial manifold by a sequence of compact partial manifolds

Let M be a partial manifold (in particular, a manifold). Since it may be covered by a denumerable number of coordinate systems, and each is compact, M may be covered by a sequence of compact partial manifolds. We wish to show how partial manifolds with certain properties may be chosen. Let a *proper half-open arc* A in a point set R mean the one-one continuous and proper image $\phi(A_0)$ of the half-line $0 \leqq x$. We shall say A runs from $\phi(0)$ to infinity if, for any compact subset B of R, there is an x_0 such that $\phi(x) \in R - B$ for $x \geqq x_0$.

The following lemma is used in the proof of the immersion theorem for open manifolds.

LEMMA 20. *Let M be an open connected partial manifold. Then there is a sequence $M_1, M_2, \cdots$ of compact partial manifolds in M such that*

(a) $M_i \subset M_{i+1} - \partial M_{i+1}$ $(i = 1, 2, \cdots)$,

(b) $M = \sum M_i$,

(c) *any point of $M - M_i$ may be joined to infinity by a half open arc which does not touch M_i.*

REMARKS. It may be shown that each M_i may be taken as connected, and such that ∂M_i is a closed manifold (not necessarily connected); but we do not need these facts here. For the present purpose, by a "partial manifold" we shall mean merely the closure of an open subset of some open manifold.

To start with, let $M_1'', M_2'', \cdots$ be the sets in M covered by a fixed set of coordinate systems, so that $M = \sum M_i''$. Set $M_1' = M_1''$. Since M_1' is compact, we may choose μ_2 so that $M_1' \subset \sum_{k=1}^{\mu_2}(M_k'' - \partial M_k'')$. Set $M_2' = \sum_{k=1}^{\mu_2} M_k''$. In general, choose μ_{i+1} so that $M_i' \subset \sum_{k=1}^{\mu_{i+1}}(M_k'' - \partial M_k'')$, and set $M_{i+1}' = \sum_{k=1}^{\mu_{i+1}} M_k''$. Then $M_i' \subset M_{i+1}' - \partial M_{i+1}'$, and $M = \sum M_i'$.

Let M_i be the set of all points p of M for which the following is not true: For each j there is an arc A joining p to a point $q \in M - M'_j$, such that $A \subset M - M'_i$. We show first that M_i is compact. If not, then there is a sequence $p_1, p_2, \cdots$ of points of M_i with no subsequence which converges in M. Since each M'_j is compact, it contains at most a finite number of the p_k; hence we may suppose the p_k chosen so that $p_j \in M - M'_j$, all j. Since M is connected, there is an arc A_j in M joining p_1 to p_j. Let q_j be the last point of A_j in M'_{i+1} for $j > i$, and let $A(p_j)$ be the arc $q_j p_j$. Since M'_{i+1} is compact and $\partial M'_{i+1}$ is closed in M'_{i+1}, there is a subsequence $q'_1, q'_2, \cdots$ of $q_1, q_2, \cdots$ converging to a point q of $\partial M'_{i+1}$. For some connected neighborhood U of q, $U \subset M - M'_i$. Choose s so that $q'_k \in U$ for $k \geqq s$. Let p'_k correspond to q'_k. For each $j > s$, $p'_j \in M - M'_j$, and from $A(p'_s)$, $A(p'_j)$, and an arc in U, we find an arc in $M - M'_i$ joining p'_s to p'_j. Hence p'_s is not in M'_i, a contradiction, proving that M_i is compact.

Clearly all boundary points of M_i are in M'_i; hence (a) holds. Since $M'_i \subset M_i$, (b) is true. To prove (c), take $p \in M - M_i$. By definition of the M_j, there is an arc A_j joining p to a point p_{i+1} in $M - M'_{i+1}$, such that $A_i \subset M - M'_i$, there is an arc A_{i+1} joining p_{i+1} to a point p_{i+2} in $M - M'_{i+2}$, such that $A_{i+1} \subset M'_{i+1}$, etc. Since $\sum M'_j = M$, these arcs give a proper half open arc A joining p to infinity, such that $A \subset M - M'_i$. Moreover, $A \subset M - M_i$; for if $q \in A \cap M_i$, then part of A gives an arc in $M - M'_i$ joining q to a point in an arbitrary $M - M_j$, contradicting the definition of M_i. This completes the proof.

24. On proper mappings

We recall the definition from [1]: The *limit set* L_f of a mapping f of a space R into a space R' is the set of points $q \in R'$ such that for some sequence $\{p_k\}$ in R, $f(p_k) \to q$, while $\{p_k\}$ has no limiting point in R. f is *proper* if $f(R) \cap L_f = 0$. Note that if R is compact, the limit set under any mapping is void. If f is proper in R, it is proper in any closed subset of R. If f maps R into a single point, f is proper if and only if R is compact.

REMARK. If f is one-one and continuous, then clearly f is proper if and only if f^{-1} is continuous.

We give first a characterization of proper mappings without use of sequences.

LEMMA 21. *A mapping f of a locally compact separable metric space R into a similar space R' is proper if and only if for each point $p \in R$ (or equally well, each self-compact subset $A \subset R$) there is a neighborhood U of $f(p)$ (or of $f(A)$) in R' and a self-compact subset B of R such that*

$$f(R - B) \cap U = 0. \tag{24.1}$$

We may suppose R is not compact, the lemma being trivial otherwise. Let $R_1, R_2, \cdots$ be self-compact subsets of R with $R_i \subset R_{i+1}$ and $R = \sum R_i$ (see the proof of Lemma 20). Suppose first that the condition in its strong form does not hold; then a self-compact subset A is given; say $A \subset R_k$. Let $\{U_i\}$ be a sequence of neighborhoods of $f(R_k)$ such that $\prod U_i = f(R_k)$. In each $R - R_{k+i}$ choose a point p_i so that $f(p_i) \in U_i$; we may suppose that the p_i are distinct. A subsequence may be chosen so that $f(p_{\lambda_i}) \to q \in f(R_k)$; thus f is not proper.

If, conversely, f is not proper, say $f(p_i) \to q = f(p)$, $\{p_i\}$ having no limit in R. For some subsequence $\{p_{\lambda_i}\}$, $p_{\lambda_i} \in R - R_i$. Since each compact subset of R is in some R_j, the condition does not hold, using p.

We state without proof:

LEMMA 22. *A mapping f of R into R' is proper if and only if antecedents of sets compact in $f(R)$ are sets compact in R.*

LEMMA 23. *A continuous proper mapping of R into R' maps sets closed in R into sets closed in $f(R)$.*

Suppose A is closed, while $f(A)$ is not closed in $f(R)$; then $q_1, q_2, \cdots$ exist in $f(A)$, $q_i \to q$, $q \in f(R) - f(A)$. Choose p_i so that $f(p_i) = q_i$. If the sequence $\{p_i\}$ had a limiting point p, then say $p_{\lambda_i} \to p$; since f is continuous, $q_{\lambda_i} = f(p_{\lambda_i}) \to f(p)$. But $q_{\lambda_i} \to q$; hence $f(p) = q$, and $q \in f(A)$, a contradiction. This shows that f is not proper.

We state without proof:

LEMMA 24. *Let f be continuous and map closed sets into closed sets; and let the antecedents of single points be finite sets of points. Then f is proper.*

The following lemma is needed in one or two places in the present paper.

LEMMA 25. *Let R and R' be locally compact separable metric spaces. Let f, mapping R into R', be continuous, proper, and locally one-one. Let A and B be closed subsets of R such that*

$$\text{(24.2)} \qquad \text{if } p \in A, \quad q \in B, \quad p \neq q, \quad \text{then } f(p) \neq f(q).$$

Then there are neighborhoods U of A and V of B such that

$$\text{(24.3)} \qquad \text{if } p \in U, \quad q \in V, \quad p \neq q, \quad \text{then } f(p) \neq f(q).$$

The proof is not very difficult; we expect to give it as an application of much more general ideas in another paper.

HARVARD UNIVERSITY

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THE CLASSIFICATION OF SPHERE BUNDLES

BY N. E. STEENROD

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1. Introduction

Whitney has introduced [13] the general notion of one space A being a fibre bundle over a second space B. It is a topological condition on a function f mapping A onto B which insures that f shall have a high degree of smoothness. It is the topological counterpart of the analytic requirement, in case A and B are differentiable manifolds, that f shall have a Jacobian of maximum rank at every point. Fibre bundles should prove of importance since they arise in many connections. The factor spaces of a Lie group form a lattice of fibre bundles (see Theorem 1). The spaces of tensors over a manifold are fibre bundles [9].

Among the simpler bundles are those for which the fibres (inverse images of points of B) are k-spheres. One of the main problems with which Whitney has concerned himself is the classification of the k-sphere bundles over a given space B.

In the present paper, this problem is reduced to a familiar problem of topology. A factor space M_l^k of the rotation group of a $(k + l + 1)$-sphere is selected, and it is proved that the equivalence classes of k-sphere bundles over a complex B are in 1-1 correspondence with the homotopy classes of maps of B in M_l^k.

In addition, the homotopy groups of M_l^k are computed for dimensions $\leqq 6$. This leads to a complete solution of the classification problem when B is a sphere of dimension $\leqq 6$.

Sections 2 and 3 contain definitions and discussion of fibre bundles and related concepts. Sections 4, 5 and 6 contain the statements of the principal results without proofs. Sections 7 and 8 are concerned with showing that covering homotopies exist in a fibre bundle. With this mechanism and its consequences at hand, the proofs of the main results are given in the remaining sections.

2. Fibre bundles

The concept of fibre bundle is somewhat complicated. There must be given three topological spaces. The first of these, denoted by A, is called the *fibre bundle*. The second, B, is called the *base space*; and the third, F, is called the *fibre*. There must likewise be given a fixed map ψ of A onto B; it is called the *projection*. (It will appear from subsequent conditions that, for each point $b \in B$, $\psi^{-1}(b)$ is homeomorphic to F.) There must also be given a fixed group G of homeomorphisms of F, so that F is a space with a geometry. Finally, there is given a family $\{N\}$ of neighborhoods covering B, and, for each N, a function ϕ_N mapping the product space $N \times F$ homeomorphically onto $\psi^{-1}(N)$ so as to satisfy two conditions. The first of these is

$$\psi\phi_N(b, y) = b \qquad \text{for } b \in N,\ y \in F.$$

For the second, define $\phi_{N,b}$ $(b \in N)$ to be the map $F \to \psi^{-1}(b)$ given by $\phi_{N,b}(y) = \phi_N(b, y)$; then, for any $b \in B$ and neighborhoods N, N' containing b, we must have

$$\phi_{N,b}^{-1}\phi_{N',b} \in G.$$

The functions ϕ are called the *coordinate functions.*

The entire structure can be pictured in A somewhat as follows. The space A is decomposed continuously into disjoint closed sets denoted generically by F and referred to as fibres. Each F possesses a geometry G. A sufficiently small neighborhood of an F can be resolved into a product space in such a way that the fibres F' neighboring F appear as parallel sections, and under the parallel displacement $F \to F'$ we have in addition $G \to G'$. The space B is the space whose points are the fibres of A, and ψ is the natural map $A \to B$ which attaches to each point of A the fibre containing it.

If the group G consists of the identity alone, then the functions ϕ_N can be amalgamated into a single function which represents A as the product space $B \times F$, and A is called the *product bundle.*

If F is a linear space and G is the full linear group, A is called a *bundle of linear spaces.* This is the case of importance to differential geometry since the tangent space of a differentiable manifold M is such a bundle. In addition each manifold of tensors of a specified type over M is a bundle of linear spaces with the base space M [see 9].

If F is a k-dimensional sphere (k-sphere) and G is the full orthogonal group, A is called a *k-sphere-bundle.* In case G is the group of rotations (i.e. the orthogonal transformations preserving orientation), A is called an *orientable* k-sphere-bundle.

If the differentiable manifold M has a Riemann metric, then, by selecting the unit sphere in the tangent space of each point, one obtains a sphere-bundle over M. A similar operation is possible in the tensor spaces of each point.

By a *change of coordinates* in the neighborhood N, we shall mean the substitution of a function ϕ'_N for ϕ_N such that the fibre bundle conditions still hold, and, in addition, for $b \in N$, we have $\phi_{N,b}^{-1}\phi'_{N,b} \in G$. If λ is a continuous map $N \to G$, we obtain a change of coordinates if we set $\phi'_{N,b} = \phi_{N,b}\,\lambda(b)$. In general we do not wish to distinguish between a bundle and one obtained from it by coordinate transformations. This leads to the notion of equivalence of bundles.

If A, A' are two bundles over the same base space B with the same fibre F and group G, they are said to be *equivalent* if there exists a homeomorphism τ of A' onto A which, for each b in B, maps the fibre $\psi'^{-1}(b)$ of A' onto the fibre $\psi^{-1}(b)$ of A in such a fashion that, for any neighborhoods N, N' containing b and attached to A, A', respectively, we have

(E) $$\phi_{N,b}^{-1}\tau\phi'_{N',b} \in G.$$

Clearly any number of coordinate changes in A merely replaces A by an equivalent bundle.

If, for example, G is the group of all homeomorphisms, the condition E is unnecessary.

The classification problem for fibre bundles is the following: *For fixed B, F and G, exhibit one and only one A from each equivalence class.*

Now let B, F be fixed and let G, G' be two groups of homeomorphisms of F such that $G \subset G'$. If A is a fibre bundle over B with fibre F and group G, then the same coordinate functions ϕ represent A as a fibre bundle over B with fibre F and group G'. This new bundle A' is distinct from A since the groups of the fibres are larger. The transformation $A \to A'$ carries equivalent bundles into such, and is therefore an operation on equivalence classes. It is to be expected in general that this correspondence of classes is a many-to-one map and need not be a mapping *onto*.

Since the identity homeomorphism of F is contained in every G, the product bundle $B \times F$ determines an equivalence class of bundles relative to B, F and G. A bundle of this class will be called a *product bundle* with little danger of confusion.

It is sometimes convenient to consider an equivalence of two bundles not possessing the same base space and fibre. Let A, B, F, G be a bundle and A', B', F', G' a second. We shall say that they are *equivalent* (in the unrestricted sense) if there exists a homeomorphism τ of A' onto A which carries fibres into fibres, inducing thereby a homeomorphism τ of B' onto B, and is such that, if $\tau(b') = b$, $b \in N$ and $b' \in N'$, then the homeomorphism $\phi^{-1}_{N,b}\tau\phi'_{N',b'}$ of F' onto F must carry the group G' onto G.

For A, A' to be equivalent, it is necessary that B, B' be homeomorphic, and that F, F' be homeomorphic in such a way that G' corresponds to G. Assuming these conditions satisfied, we cannot specify these correspondences in advance unless we wish to consider the restricted equivalence defined earlier.

On the other hand, two bundles A, A' with the same B, F, G may be equivalent in the unrestricted sense but not in the restricted sense. Of course, in such a case, τ would of necessity induce a non-trivial transformation of B, or of F or of both. Naturally there are fewer equivalence classes in the unrestricted sense.

In our main problem equivalence will be used only in the restricted sense.

3. Analogy with group extensions

We digress for a moment to remark the analogy between a fibre bundle and a group extension. In the group case F and G coalesce into the group G, B is likewise a group, and A is an extension of G by B (i.e. G is an invariant subgroup of A and $A/G = B$).

The equivalence of two extensions parallels the equivalence of two bundles; in the first case τ is isomorphic, in the second, homeomorphic.

The problem of classifying bundles with fixed B, F and G is the analog of determining all extensions of the group G by the group B.

We shall see that these ideas come together quite naturally when A, B and G are topological groups.

4. Summary of principal results

If R is a topological group and K is a closed subgroup, the symbol R/K is used to denote the set of left cosets of K in R. The *natural map* $R \to R/K$ attaches to each r in R the coset rK. A set of cosets is an *open set* of R/K if their point set sum is an open set of R. With this definition of open set, R/K is a topological space, and the natural map is continuous [see 8, Ch. 3]. If the elements of a coset of K are all multiplied on the left by a fixed element of R, the resulting set is also a left coset of K. In this way R is realized as a group of homeomorphisms of R/K. Of course R may not operate effectively in R/K; the effective group is R/K' where K' is the intersection of all subgroups conjugate to K.

It is well-known that, if R is a Lie group, then R/K is an analytic manifold, and the natural map has Jacobian of maximum rank at every point. Even more, we have

THEOREM 1. *If R is a Lie group, and H, K are closed subgroups with $H \subset K$, then, with respect to the natural map, R/H is a fibre bundle over R/K with fibre $F = K/H$ subject to the group $G = K$ of left translations of K/H.*

Now let R be the group of rotations (i.e. the orthogonal maps preserving orientation) of a sphere S of dimension $k + l + 1$. In S we select a fixed great k-sphere S^k. Let S^l be the great sphere of S orthogonal to S^k. A rotation of S^k (S^l) becomes a rotation of S if we let it act as the identity on S^l (S^k). In this way the groups of rotations $R(S^k)$, $R(S^l)$ of S^k, S^l, respectively, are subgroups of R. Since they commute with one another, we may regard their direct product

$$\tilde{R}(S^k, S^l) = R(S^k) \times R(S^l)$$

as a subgroup of R. It is contained in the group $R(S^k, S^l)$ of all elements of R which map S^k on itself. The latter is larger since it contains the rotations of S which reverse orientations of both S^k and S^l. Since any two such differ by an element of $\tilde{R}(S^k, S^l)$, it follows that $R(S^k, S^l)$ has just two components and $\tilde{R}(S^k, S^l)$ is the component of the identity. We shall make extensive use of the left coset spaces

$$M_l^k = R/R(S^k, S^l), \qquad \tilde{M}_l^k = R/\tilde{R}(S^k, S^l).$$

From Theorem 1, we obtain

$\tilde{M}_l^k$ is a two-fold covering of M_l^k.

Two rotations r, r' lie in the same left coset of $R(S^k, S^l)$ if and only if $r(S^k) = r'(S^k)$. Since any great k-sphere of S is the image $r(S^k)$ for a suitable r, the correspondence $r \to r(S^k)$ between the great k-spheres and the left cosets of $R(S^k, S^l)$ is 1-1. Therefore

M_l^k is the space whose points are the great k-spheres of a $(k + l + 1)$-sphere.[1]

[1] A simple duality shows that the bundle R over M_l^k is equivalent to the bundle R over M_k^l. In particular M_l^k and M_k^l are homeomorphic. We might therefore regard M_l^k as the space of great l-spheres of a $(k + l + 1)$-sphere. We adhere to the first interpretation throughout the paper.

By a similar argument, we obtain

$\tilde{M}_l^k$ is the space whose points are the oriented great k-spheres of a $(k + l + 1)$-sphere.

We might equally well regard M_l^k as the space of $(k + 1)$-planes passing through the origin of a $(k + l + 2)$-space.

It is worth noting that M_l^0 is projective $(l + 1)$-space and $\tilde{M}_l^0$ is an $(l + 1)$-sphere.

DEFINITION. To a map g of a topological space B into M_l^k we attach a space $A(g)$ as follows: $A(g)$ is the subset of $B \times S^{k+l+1}$ consisting of those pairs (b, s) such that s is a point of the k-sphere corresponding to $g(b)$. Define $\psi(b, s) = b$, so that ψ maps $A(g)$ onto B.

It is clear that $\psi^{-1}(b)$ in $A(g)$ is the k-sphere $b \times g(b)$. The main results hinging on the function $A(g)$ are embodied in the following theorems.[2]

THEOREM 2. *For any map g of B in M_l^k, the space $A(g)$ is a k-sphere bundle over B with projection ψ.*

THEOREM 3. *If B is compact and g, g' are homotopic maps of B in M_l^k, then the bundles $A(g)$, $A(g')$ are equivalent.*

THEOREM 4. *If B is a complex,[3] A a k-sphere bundle over B, and $l \geqq \dim B - 1$, then there is a map g of B in M_l^k such that $A(g)$ is equivalent to A.*

THEOREM 5. *If B is a complex,[3] $l \geqq \dim B$, and g, g' are two maps of B in M_l^k such that $A(g)$, $A(g')$ are equivalent, then g, g' are homotopic.*

As an immediate consequence of Theorems 2 to 5, we have

COROLLARY. *The problem of classifying the k-sphere bundles over a complex B is equivalent to the problem of enumerating the homotopy classes of maps of B in M_l^k for any $l \geqq \dim B$.*

5. The homotopy structure of M_l^k

It should be observed that, if g maps B into a point of M_l^k, then $A(g)$ is a product bundle $B \times S^k$. From this and Theorem 3, it follows that inessential maps of B in M_l^k determine product bundles. As a corollary, if B is contractible on itself to a point, then any sphere bundle over B is a product bundle.

To obtain bundles which are not product bundles one must therefore study spaces B which are not contractible. The simplest such are the spheres of various dimensions. This leads to the problem of classifying the maps of spheres in M_l^k, and this in turn to the determination of the homotopy groups $\pi_i(M_l^k)$. Furthermore an inspection of existing results on the number of homotopy classes of maps of one space in another [5] reveals that the homotopy groups of the latter space play a fundamental role. Therefore our immediate objective is to obtain such information as we can about the groups $\pi_i(M_l^k)$.

Much is already known concerning the homotopy groups of the orthogonal groups [2; 12]. Since M_l^k is intimately related to several such groups, the obvious procedure is to examine this connection more closely. This leads to the following results.

[2] Theorems 2 and 4 have been stated by Whitney in a different but equivalent form (see [15, §8]).

[3] The complex B may be infinite but is required to be locally finite.

THEOREM 6. *If k or l is positive,*[4] *the fundamental group of M_l^k is cyclic of period* 2. *If $R(S^k)$ is the group of rotations of S^k, and if it is considered as a subgroup of R, then there is a natural isomorphism*

$$\pi_i(M_l^k) \cong \pi_{i-1}(R(S^k)) \qquad 2 \leqq i \leqq l.$$

The restriction $i \leqq l$ is of no significance in the classification problem since l may be chosen as large as convenient.

Consider now the classification problem for $B = i$-sphere. An element of $\pi_i(M_l^k)$ is a class of maps of B in M_l^k which map a fixed reference point b_0 of B into another such point x_0 of M_l^k, and any two maps of the class are homotopic within the class. If C_i denotes the set of homotopy classes of maps B in M_l^k, then each element of $\pi_i(M_l^k)$ is contained in some element of C_i; so that a map $\pi_i \to C_i$ is at hand. Since M_l^k is arcwise connected, any map of B in M_l^k is homotopic to one mapping b_0 into x_0. Thus C_i is the image of π_i.

The algebraic background of the correspondence $\pi_i \to C_i$ has been given by Eilenberg [5, pp. 63–64] as follows. A closed path beginning and ending at x_0 determines an automorphism of π_i. The effect of the automorphism on any i-sphere with b_0 mapped into x_0 is obtained by deforming the sphere in M_l^k so that b_0 describes the closed path. The automorphism depends only on the homotopy class of the closed path. Indeed, the fundamental group $\pi_1(M_l^k)$ appears as a group of automorphisms of $\pi_i(M_l^k)$. The principal result that concerns us is the following:

Two elements of $\pi_i(M_l^k)$ belong to the same homotopy class of maps of an i-sphere in M_l^k if and only if there is an element of $\pi_1(M_l^k)$ mapping one into the other.

Whenever each element of π_1 induces the identity automorphism in π_i the space is said to be *simple* in the dimension i. In this case the map $\pi_i \to C_i$ is 1-1.

We are able to establish

THEOREM 7. *If k and l are even, then M_l^k is simple in all dimensions. In any case M_l^k is simple in all dimensions* $\leqq$ Min (k, l). *If $l \geqq 2$, then M_l^1 is not simple in the dimension* 2; *for $\pi_2(M_l^1)$ is infinite cyclic and the non-trivial element α of $\pi_1(M_l^1)$ reverses sign in $\pi_2(M_l^1)$.*

We have not been able to decide the question of simplicity in the remaining cases except for trivial cases where $\pi_i = 0$ or is cyclic of period 2.

6. Sphere bundles over spheres

Combining the results of the preceding section with the known facts [2; 12] about the groups $\pi_{i-1}(R(S^k))$, we can obtain numerous special results concerning the number of k-sphere bundles over a sphere B of dimension $\leqq 6$. We state some of these in this section.

Since $\pi_1(M_l^k)$ is cyclic of period 2, we have

I. *The k-sphere bundles over S^1 are of two types: the product bundle, and the generalized Klein bottle.*

[4] The case $k = l = 0$ is trivial since M_0^0 is a 1-sphere.

The term *Klein bottle* is justified by the fact that, if S^k is carried around S^1, it comes back with orientation reversed. This is proved by noting that the non-trivial element α of $\pi_1(M_l^k)$ is covered by a path in R joining the identity to an element of $R(S^k, S^l)$ which reverses the orientation of S^k.

Since $\pi_2(M_l^1) \cong \pi_1(R(S^1))$ is infinite cyclic,

II. *There are an infinity of* 1*-sphere bundles over* S^2.

Since the automorphism α is a reversal of sign, an element of $\pi_2(M_l^1)$ defines the same 1-sphere bundle as does its negative. We can thus set up a natural correspondence between the integers $n = 0, 1, 2, \cdots$ and the 1-sphere bundles over S^2. Let A_n denote the bundle corresponding to n. It is clear the A_0 is the product bundle.

It is well known that, if S^3 = group of unit quaternions, and S^1 = subgroup of complex numbers of absolute value 1, then S^3/S^1 is homeomorphic to S^2. In this way S^3 is a 1-sphere bundle over S^2 (see Theorem 1). It is a reasonable conjecture that this is the bundle A_1.

If G_m is the subgroup of S^1 generated by $\exp(2\pi i/m)$, by Theorem 1, S^3/G_m is a bundle over $S^3/S^1 = S^2$. The fibre is S^1/G_m which is again a 1-sphere. In this way S^3/G_m is a 1-sphere bundle over S^2. It is reasonable to conjecture that it is the bundle A_m. Note that S^3/G_m is the lens space corresponding to the pair of integers $(m, 1)$.

Since $\pi_1(R(S^k))$ is cyclic of period 2 for $k \geqq 2$, we have

III. *If* $k \geqq 2$, *there is just one* k*-sphere bundle over* S^2 *other than the product bundle.*

Since $\pi_2(R(S^k)) = 0$ for any k, we have

IV. *A* k*-sphere bundle over* S^3 *is always a product bundle.*

Since $\pi_{n-1}(R(S^1)) = 0$ for $n \geqq 3$, we have

V. *If* $n \geqq 3$, *a* 1*-sphere bundle over* S^n *is always a product bundle.*

Since $\pi_3(R(S^k))$ contains an infinity of elements for $k > 1$, we have

VI. *If* $k > 1$, *there are an infinity of* k*-sphere bundles over* S^4.

Since $\pi_4(R(S^k)) = 0$ for $k \geqq 5$, and $\pi_5(R(S^k)) = 0$ for $k \neq 5$, we have

VII. *A* k*-sphere bundle over* S^5 (S^6) *is always a product bundle if* $k \geqq 5 (k \neq 5)$.

One may obtain additional information by examining the results of [2; 12].

7. Factor spaces of groups

Since we must deal extensively with the factor spaces of the rotation group R, a few general remarks about factor spaces are in order.

DEFINITION. Let R be a topological group and K a closed subgroup. A function ϕ defined in a neighborhood N of the point K of the left coset space R/K with values in R is called a *slicing function* for K if (1) ϕ is continuous, (2) $\phi(b)$ is an element of b for each b in N, (3) $\phi(K)$ is the identity element of R. We shall say that ϕ is *invariant* under the subgroup H of R if $hKh^{-1} = K$ and $\phi(hbh^{-1}) = h\phi(b)h^{-1}$ for each b in N and h in H.

A slicing function is a continuous cross-section through the identity of the family of left cosets of K neighboring K. It is invariant if the cross-section is

mapped on itself when conjugated by an element of H. In practice, H will be a subgroup of K.

THEOREM 8. *If K is a closed subgroup of the Lie group R, then there exists a slicing function ϕ for K. If H is a compact subgroup of R such that $hKh^{-1} = K$ for each h in H, then ϕ may be chosen invariant under H.*

PROOF. Choose a system of canonical coordinates of the first kind for a neighborhood of the identity e of R [see 8, p. 187]. A sufficiently small neighborhood of e in K is a linear subspace in these coordinates [8, p. 196, 203]. By means of a linear transformation, it can be arranged that the subspace K is defined by equations $x^1 = x^2 = \cdots = x^r = 0$ where $(x) = (x^1, \cdots, x^m)$ are the coordinates of the element x of R. If the elements x, y are near enough to e, then the coordinates of $z = xy$ are differentiable functions of those of x and y: $z^i = f^i(x, y)$. The system of equations

$$\text{(A)} \qquad f^i(x^{-1}, y) = 0 \qquad (i = 1, \cdots, r), \qquad y^j = 0 \qquad (j = r + 1, \cdots, m)$$

reduces to the system $y^i = 0$ $(i = 1, \cdots, m)$ when $x = e$. Thus the Jacobian of (A) in the y^i's is non-singular for x, y near e. It follows from the implicit function theorem that there exists a system of continuous functions $\phi^i(x)$, defined for x in a neighborhood of e, such that (A) becomes an identity in x when y^i is replaced by $\phi^i(x)$. The system (A) states that y lies in the intersection of the left coset xK with the coordinate subspace complementary to K. The uniqueness part of the implicit function theorem assures us that there is just one such intersection near to e for x near enough to e. Therefore, for such x's, $\phi(x)$ is constant along the coset xK. Thus ϕ is a function of the coset alone and is therefore the desired slicing function.

To prove the second part of the theorem, observe first that the operation of conjugation of R by a fixed element of R is a linear transformation of the canonical coordinates of the first kind [8, p. 280]. Since the subgroup H is compact, there is a linear transformation to new coordinates in which conjugation by an element of H is an orthogonal transformation.[5] Since K is invariant under H, the linear subspace orthogonal to K is likewise invariant. These new coordinates are canonical, and may be chosen so that K is a coordinate subspace. Using these coordinates, we define ϕ as above. Then, for h in H, both $\phi(hxh^{-1})$ and $h\phi(x)h^{-1}$ lie in the intersection of the left coset $(hxh^{-1})K$ and the coordinate plane orthogonal to K. Since this intersection consists of just one point near e when x is near e, we have $\phi(hxh^{-1}) = h\phi(x)h^{-1}$ as desired.

Theorem 1 is an immediate consequence of Theorem 8 and the following

THEOREM 1′. *If R is a topological group, K a closed subgroup of R which admits a slicing function ϕ, and H a closed subgroup of K, then, with respect to the natural map, R/H is a fibre bundle over R/K with fibre $F = K/H$ subject to the group $G = K$ of left translations of K/H.*

[5] In the rotation group R with which we are mainly concerned, the Cayley coordinates have this property [11, p. 56].

PROOF. Let ψ denote the natural map $R/H \to R/K$. Let N denote the neighborhood of the point K of R/K in which ϕ is defined. The left translation of N by r in R is denoted by N_r. The totality of these translations covers R/K. For each N_r, we define a map ϕ_r of $N_r \times K/H$ into R/H as follows

$$\phi_r(b, c) = [r\phi(r^{-1}b)]c, \qquad b \in N_r, c \in K/H.$$

Since b is in N_r, $r^{-1}b$ is in N, so that $\phi(r^{-1}b)$ is defined. Since $K/H \subset R/H$, $c \in K/H$ is an element of R/H, and $\phi_r(b, c)$ is the left translation of c by the element $r\phi(r^{-1}b)$ of R. It is clear that ϕ_r is continuous.

Since $\phi(r^{-1}b)$ is a point of the coset $r^{-1}b$, $r\phi(r^{-1}b)$ is a point of the coset b. If c be regarded as a subset of K, it follows that $\phi_r(b, c)$ is a subset of b; and therefore $\psi\phi_r(b, c) = b$.

For an element a in $\psi^{-1}(N_r)$, let $\zeta_r(a)$ be the left translation of a by $[r\phi(r^{-1}\psi(a))]^{-1}$. It follows that $\zeta_r\phi_r(b, c) = c$, and $\phi_r(\psi(a), \zeta_r(a)) = a$. The existence of the continuous functions ψ and ζ_r proves that ϕ_r is a homeomorphism of $N_r \times K/H$ onto $\psi^{-1}(N_r)$.

Finally if b is in both N_r and $N_{r'}$, the map $\phi_{r,b}^{-1}\phi_{r',b}$ of K/H on itself is given by

$$c' = [r'\phi(r'^{-1}b)]^{-1}[r\phi(r^{-1}b)]c$$

which is clearly a left translation of K/H by an element of K.

8. Covering homotopies

Suppose ψ is a continuous map of A into B, f a continuous map of X into A, and $h(x, t)$, $x \in X$, $0 \leqq t \leqq 1$, a homotopy in B of the map ψf of X into B (i.e. $h(x, 0) = \psi f(x)$). A homotopy $g(x, t)$ of f in A is said to *cover* the homotopy $h(x, t)$ if $\psi g(x, t) = h(x, t)$ for all x, t. Clearly, if a homotopy $g(x, t)$ of f be given, then $g(x, t)$ covers the homotopy $\psi g(x, t)$. Of more significance is the matter of constructing a $g(x, t)$ for a given $h(x, t)$.

The covering homotopy g of h is said to be *stationary* with h, if, whenever $h(x_0, t)$ is constant in a t-interval $t' \leqq t \leqq t''$, then $g(x_0, t)$ is likewise constant in that interval (i.e. whenever x_0 remains at rest under h, it is likewise at rest under g).

In a recent paper by Hurewicz and the author [7] the question of the existence of covering homotopies is considered in some detail. The property "A is a fibre space over B relative to ψ" is introduced, and it is proved that, in the presence of this property, there exists a g covering a given h which is stationary with h.

The notions of fibre space and fibre bundle do not coincide, there are fibre spaces that are not fibre bundles. The fibres of a fibre space need not be pairwise homeomorphic, although they do belong to the same homotopy type. Whether every fibre bundle is a fibre space is not yet determined. What concerns us here is to establish the existence of covering homotopies for fibre bundles, for we shall make extensive use of the consequences.

THEOREM 9[6]. *If A is a fibre bundle over B, X a compact space, f a continuous*

[6] The essential content of this theorem has been stated by Ehresmann and Feldbau [4].

map $X \to A$, and $h(x, t)$ a homotopy of the map ψf of X in B, then there exists a covering homotopy $g(x, t)$ which is stationary with $h(x, t)$.

PROOF. For each b in B we select a pair of neighborhoods U, V of b such that $\bar{U} \subset V$ and $\bar{V}$ is contained in some coordinate neighborhood N. Since $X \times I$ is compact (I is the unit t-interval), there exists a $\delta > 0$ such that, for any point x of X and interval $I' \subset I$ of length $<\delta$, the image of $x \times I'$ under h is contained wholly in some member of the family $\{U\}$. Let $0 = t_0 < t_1 < \cdots < t_n = 1$ be a partition of I such that $t_{i+1} - t_i < \delta$. We shall suppose that $g(x, t)$ has been constructed for all x, t such that $t \leqq t_k$, and proceed to show that g may be extended as required over the interval I': $t_k < t \leqq t_{k+1}$.

Since, for each x, there exists a U containing the image of $x \times I'$ under h, it follows by continuity that there is a neighborhood W of x such that h maps $W \times I'$ into some U. Since X is compact, we may select a finite covering by these neighborhoods: $W_1, \cdots, W_m$. Denote the U, V pair such that $W_i \times I' \to U$ by U_i, V_i.

By the Urysohn lemma, there exist a continuous, real-valued function $u_i(b)$ defined on B such that $u_i = 1$ for b in $\bar{U}_i$, $u_i = 0$ for b outside V_i, and $0 \leqq u_i \leqq 1$. Define

$$\tau_j(x) = t_k + (t_{k+1} - t_k) \operatorname{Max}_{i \leqq j} u_i(h(x, t_k)).$$

Clearly τ_j is continuous, $t_k \leqq \tau_j \leqq t_{k+1}$, $\tau_j(x) \leqq \tau_{j+1}(x)$, and $\tau_m(x) = t_{k+1}$. Define $\tau_0(x) = t_k$. Then $g(x, t)$ is defined for $t \leqq \tau_0(x)$. We shall suppose, inductively, that $g(x, t)$ has been properly constructed for $t \leqq \tau_j(x)$, and show that it may be extended to $t \leqq \tau_{j+1}(x)$.

Denote by T the set of pairs (x, t) such that $\tau_j(x) < t \leqq \tau_{j+1}(x)$. It is over this set that we must extend $g(x, t)$. It follows from the definition of the functions τ that h maps the closure $\bar{T}$ of T into $\bar{V}_{j+1}$ which is contained in some coordinate neighborhood N with coordinate function ϕ. Since ϕ is a homeomorphism of $N \times F$ with $\psi^{-1}(N)$, there exists a map ζ of $\psi^{-1}(N)$ into F such that $\zeta\phi(b, c) = c$ for each c in F. Since $(x, t) \in \bar{T}$ implies $\psi g(x, \tau_j(x)) = h(x, \tau_j(x))$ is in N, the function $\zeta g(x, \tau_j(x))$ is defined. We can therefore define

$$g(x, t) = \phi(h(x, t); \zeta g(x, \tau_j(x))) \qquad \text{for } (x, t) \text{ in } T.$$

It follows from the properties of ϕ that $\psi g(x, t) = h(x, t)$, and that g is stationary with h. To see that g has been continuously extended, we note that the expression for g is defined and continuous for (x, t) in $\bar{T}$, and, for (x, t) in $\bar{T} - T$ it agrees with the values of g already assigned.

This completes the general step in the secondary induction. Since $\tau_m(x) = t_{k+1}$, the completion of the secondary induction establishes the general step of the primary induction. This completes the proof.

As an immediate consequence of Theorem 9 and a theorem of R. H. Fox[7], we have

[7] Fibre spaces II, Bull. Amer. Math. Soc. 49 (1943).

THEOREM 10. *If A is a compact fibre bundle over B, and B is an absolute neighborhood retract, then A is a fibre space over B.*

In the case of factor spaces of groups we can establish directly the fibre space nature of the fibre bundles.

THEOREM 11. *Let R be a topological group and H, K closed subgroups with $H \subset K$. If H admits a slicing function, and K admits a slicing function invariant under H, then R/H is a fibre space over R/K with the natural map as projection.*

PROOF. Let $\phi(b)$ be the slicing function for K which is invariant under H. Let η be the natural map $R \to R/H$, and ψ the natural map $R/H \to R/K$. For any pair (a, b) such that a is in R/H and b is in the left translation to $\psi(a)$ of the domain N of definition of ϕ, we define the slicing function

$$\phi(a, b) = \eta(r\phi(r^{-1}b))$$

where r has been chosen arbitrarily in $\eta^{-1}(a)$. That ϕ is independent of the choice of r in $\eta^{-1}(a)$ follows from the invariance of ϕ under H; for

$$(rh)\phi((rh)^{-1}b) = r\phi(r^{-1}bh^{-1})h = r\phi(r^{-1}b)h.$$

To prove that ϕ is continuous in (a, b), it suffices to note that, since H admits a slicing function, we may choose r in $\eta^{-1}(a)$ as a continuous function of a neighboring a particular choice r_0 in $\eta^{-1}(a_0)$.

COROLLARY. *If R is a Lie group, K a closed subgroup, and H a compact subgroup of K, then R/H is a fibre space over R/K with the natural map as projection.*

9. Proofs of Theorems 6 and 7

We first prove that $\tilde{M}_l^k$ is simply-connected. Let a_0 be the identity element of R, and let b_0 be its projection in $\tilde{M}_l^k$. Let $g(x)$ be a continuous map of the interval $0 \leqq x \leqq 1$ in $\tilde{M}_l^k$ with $g(0) = g(1) = b_0$. If it is not already the case, one can obtain by a simple homotopy that $g(x) = b_0$ for $\frac{1}{2} \leqq x \leqq 1$. Since g may be regarded as a homotopy of the image of a_0, it follows from the existence of a covering homotopy that there is a function $f(x)$ defined for $0 \leqq x \leqq \frac{1}{2}$ with values in R such that $f(0) = a_0$, and the projection of $f(x)$ is $g(x)$. This last implies that $f(\frac{1}{2})$ is in $\tilde{R}(S^k, S^l)$ which, being connected, admits a path $f(x)$, $\frac{1}{2} \leqq x \leqq 1$, from $f(\frac{1}{2})$ to a_0. Then the projection of $f(x)$ is $g(x)$ for $0 \leqq x \leqq 1$. By assumption k or l is positive, say l. It is proved in [7, Th. 5] that a closed path f in R is contractible into $R(S^l)$. The image in $\tilde{M}_l^k$ of this contraction of f provides the desired contraction of g into b_0.

Since $\tilde{M}_l^k$ is simply-connected and is a two-fold covering of M_l^k, it follows that the fundamental group of M_l^k is cyclic of period 2. The non-trivial element of this group is denoted by α.

Our procedure in proving Theorems 6 and 7 is to exhibit a set of four natural isomorphisms. Their combination will be the isomorphism of Theorem 6. Under these isomorphisms the automorphism α of $\pi_i(M_l^k)$ is transformed into an automorphism of each of the groups, likewise denoted by α. In each case, the automorphism will be exhibited in a new form.

The first of these isomorphisms is

$$\pi_i(M_l^k) \cong \pi_i(\tilde{M}_l^k) \qquad i \geqq 2 \tag{A}$$

that always exists [7, Cor. 3] between the groups of a space and its covering space. and is induced by the covering map $\xi\colon \tilde{M}_l^k \to M_l^k$.

It is well known that the fundamental group of a space X is isomorphic in a natural way to the group of covering transformations of its universal covering space $\tilde{X}$. Since $\tilde{X}$ is simple in all dimensions, a homeomorphism of $\tilde{X}$ on itself induces an automorphism of $\pi_i(\tilde{X})$. In this way $\pi_1(X)$ appears as a group of automorphisms of $\pi_i(\tilde{X})$. Thus $\pi_1(X)$ operates in both $\pi_i(X)$ and $\pi_i(\tilde{X})$. Eilenberg has shown [6, p. 171] that these operations commute with the natural isomorphism $\pi_i(X) \cong \pi_i(\tilde{X})$.

The covering transformation α of $\tilde{M}_l^k$ is obtainable as follows. Let r_0 be any rotation of S which maps S^k on itself with orientation reversed. It will then map S^l on itself with orientation reversed. Therefore r_0 lies in the component of $R(S^k, S^l)$ other than the component of the identity $\tilde{R}(S^k, S^l)$. If a left coset of $R(S^k, S^l)$ be multiplied on the *right* by r_0, the two components of the coset are interchanged. As an operation on the left cosets of $\tilde{R}(S^k, S^l)$, it is precisely the covering transformation α.

Left translation of $\tilde{M}_l^k$ by any r in R is homotopic to the identity; for R is arcwise connected, and a path in R from r to a_0 provides the 1-parameter family of left translations. (Note that right translation of $\tilde{M}_l^k$ by r has no meaning unless r is in $R(S^k, S^l)$; for otherwise the image of a left coset of $\tilde{R}(S^k, S^l)$ is not a left coset.) Therefore right translation by r_0 followed by left translation by r_0^{-1} (i.e. conjugation by r_0) is a map T of $\tilde{M}_l^k$ homotopic to the covering transformation α. It induces, therefore, the automorphism α in $\pi_i(\tilde{M}_l^k)$. It has the advantage that it leaves fixed the point b_0; for $r_0^{-1}\tilde{R}(S^k, S^l)r_0 = \tilde{R}(S^k, S^l)$. We choose b_0 as the base point for $\pi_i(\tilde{M}_l^k)$.

Suppose, for the moment, that both k and l are even. In this case, the dimension of S is odd so that the antipodal transformation of S is in R and may be chosen as r_0. Since r_0 commutes with each element of R (r_0 is represented by the scalar matrix -1 of order $k + l + 2$), it follows that conjugation by r_0 is the identity in $\tilde{M}_l^k$. This proves the first part of Theorem 7.

We introduce now a space of importance in subsequent sections as well as the present one. As in §4, $R(S^l)$ is a subgroup of R, and we define

$$N_l^k = R/R(S^l)$$

using left cosets. It is easy to see that two elements r, r' of R lie in the same left coset of $R(S^l)$ if and only if they coincide on S^k (i.e. $r(y) = r'(y)$ for each y in S^k). On the other hand, any orthogonal map of S^k in S can by extended to all of S so as to be a rotation of S. In this way N_l^k *is the space of all orthogonal maps of* S^k *in* S. It will be so regarded in the sequel.

Since $R(S^l) \subset \tilde{R}(S^k, S^l)$, we have the natural maps

$$\eta\colon N_l^k \to \tilde{M}_l^k, \qquad \zeta\colon R \to N_l^k.$$

By Theorem 1, each space is a fibre bundle over its image with the natural map as projection.

We note that ζ maps $R(S^k)$ homeomorphically onto a subset $R^*(S^k)$ which is the fibre of N_l^k over the point b_0. The point $n_0 = \zeta(a_0)$ is taken as base point for homotopy groups in N_l^k.

Since $R^*(S^k)$ is the fibre over b_0, the projection η induces, by [7, Th. 2], a natural isomorphism

$$\pi_i(\tilde{M}_l^k) \cong \pi_i(N_l^k, R^*(S^k)). \tag{B}$$

The element r_0, chosen as above, has the property $r_0^{-1}R(S^l)r_0 = R(S^l)$. Therefore conjugation of R by r_0 permutes the left cosets of $R(S^l)$, and yields thus a transformation T' of N_l^k. Since conjugation in R preserves the inclusion relations among the subsets of R, we have $\eta T' = T\eta$. It is also clear that $T'(R^*(S^k)) = R^*(S^k)$, and $T'(n_0) = n_0$. We have thus proved that the automorphism α of $\pi_i(N_l^k, R^*(S^k))$ is that induced by T'.

An element of $\pi_i(N_l^k, R^*(S^k))$ is represented by a map of an i-cell in N_l^k with boundary mapped into $R^*(S^k)$ and a fixed reference point of the boundary into n_0. The correspondence between the i-cell and its boundary has been shown [12, Th. 1] to induce a homomorphism

$$\pi_i(N_l^k, R^*(S^k)) \to \pi_{i-1}(R^*(S^k)). \tag{C}$$

Since the image under T' of the boundary of a cell is the boundary of its image, the operations of T' in the two groups of (C) commute with the homomorphism.

We shall prove that (C) is an isomorphism if $i \leqq l$. As a first step, we prove

LEMMA. *$\pi_i(N_l^k) = 0$ if $i \leqq l$.*

Since R is a fibre bundle over N_l^k with fibre $R(S^l)$, we have by [7, Th. 2] that the projection ζ induces an isomorphism

$$\pi_i(N_l^k) \cong \pi_i(R, R(S^l)).$$

In [7, proof of Th. 5], it is shown that, if $i \leqq l$, the image of an i-cell in R with boundary mapped into $R(S^l)$ is contractible into $R(S^l)$ leaving its boundary fixed. But this is equivalent to $\pi_i(R, R(S^l)) = 0$, $i \leqq l$, and the lemma is proved.

An element u of the kernel of the homomorphism (C) is represented by a map of an i-cell with the property that its boundary is contractible in $R^*(S^k)$ to n_0. If this homotopy of the boundary be extended to the cell, we find that u is represented by a map of an i-cell with boundary mapped into n_0. By the lemma, if $i \leqq l$, the image of the i-cell may be contracted into n_0 leaving the boundary fixed. Therefore, the kernel of the homomorphism (C) is zero if $i \leqq l$.

An element v of $\pi_{i-1}(R^*(S^k))$ may be represented by a map of an $(i-1)$-sphere in $R^*(S^k)$. If $i - 1 \leqq l$, we have, by the lemma, that the map may be extended continuously over an i-cell with the given $(i-1)$-sphere as boundary. We obtain in this way an element u of $\pi_i(N_l^k, R^*(S^k))$ whose image under (C) is v. Therefore (C) is an isomorphism for $i \leqq l$ and a homomorphism *onto* for $i = l + 1$.

The final isomorphism

(D) $$\pi_{i-1}(R^*(S^k)) \cong \pi_{i-1}(R(S^k))$$

is that induced by ζ which, as observed before, is a homeomorphism of $R(S^k)$ onto $R^*(S^k)$. The combination of the isomorphisms (A), (B), (C) and (D) is the isomorphism stated in Theorem 6.

It is clear that the transformation T' of $R^*(S^k)$ corresponds under ζ to the operation T'' of conjugating $R(S^k)$ by r_0. Since each element of $R(S^k)$ acts as the identity on S^l, it follows that T'' is completely determined by the transformation r_0 restricted to S^k. Thus we have

THEOREM 7'. *Under the isomorphism of Theorem* 6, *the automorphism* α *of* $\pi_i(M_l^k)$ *is transformed into the automorphism of* $\pi_{i-1}(R(S^k))$ *induced by conjugating* $R(S^k)$ *by any orientation reversing orthogonal map of* S^k.

Choose a fixed great $(k-1)$-sphere S^{k-1} on S^k. As before, we may regard $R(S^{k-1})$ as a subgroup of $R(S^k)$. Choose the orthogonal map of Theorem 7' to be the reflection of S^k through the k-plane of S^{k-1}. Since S^{k-1} remains fixed, $R(S^{k-1})$ remains fixed under the conjugation of $R(S^k)$. In [7, Th. 5], it is shown that the identity map of $R(S^{k-1})$ in $R(S^k)$ induces an isomorphism between $\pi_{i-1}(R(S^{k-1}))$ and $\pi_{i-1}R(S^k)$ for $i \leqq k-1$ and a homomorphism onto if $i = k$. The last two statements imply that, if $i \leqq k$, the automorphism of $R(S^k)$ induces the identity automorphism of $\pi_{i-1}(R(S^k))$. This completes the proof that M_l^k is simple in dimensions $i \leqq k, l$.

Consider now the case $k = 1$, $i = 2$ and $l \geqq 2$. In this case S^k and $R(S^k)$ are both circles. It is easily verified that the conjugation of a rotation by a reflection reverses the sense of the rotation. Therefore the automorphism α of $\pi_1(R(S^k))$ is the reversal of sign. Thus, by Theorem 7', M_l^1 is not simple in the dimension 2.

REMARK: Eilenberg has shown [6, p. 175] that the projective space M_0^k of dimension $k+1$ is simple or not in the dimension $k+1$ according as $k+1$ is odd or even.

10. Proof of Theorem 2

Since the space N_l^k of §9 is a fibre bundle over M_l^k, we may choose a family $\{U\}$ of coordinate neighborhoods covering M_l^k, and a family $\{\phi_U\}$ of coordinate functions. The fibre of this bundle is $R(S^k, S^l)/R(S^l)$. Choose a fixed element u_0 of the fibre and define

$$\zeta_U(x) = \phi_U(x, u_0) \qquad \text{for } x \in U.$$

The function ζ_U provides a continuous cross-section of the fibres over U.

Corresponding to a map g of B in M_l^k, we define the family $\{N\}$ of coordinate neighborhoods in B to be the inverse images in B of the neighborhoods $\{U\}$ of M_l^k. We have seen in §9 that each element n of N_l^k is an orthogonal map of S^k into S. Let us agree to denote the image of the point y of S^k under the map n by $n \circ y$. We can then define the coordinate functions for $A(g)$ by

$$\phi_N(b, y) = (b, \zeta_U(g(b)) \circ y), \qquad b \in N = g^{-1}(U), \qquad y \in S^k.$$

Since $\zeta_U(g(b))$ maps S^k on the k-sphere $g(b)$, it follows that ϕ_N maps $N \times S^k$ on the part of $A(g)$ lying over N in such a way that $\psi\phi_N(b, y) = b$. If $g(b)$ is in two neighborhoods U, U', then $\zeta_U(g(b))$ and $\zeta_{U'}(g(b))$ map S^k orthogonally into the k-sphere $g(b)$. One followed by the inverse of the other is surely an orthogonal map of S^k on itself.

11. Proof of Theorem 3

For each point x of M_l^k, let X denote the great k-sphere of S corresponding to x as indicated in §4. In particular, let x_0 be the point of M_l^k corresponding to the reference sphere S^k.

By Theorem 8, there exists a slicing function ϕ for $R(S^k, S^l)$ invariant with respect to $R(S^k, S^l)$. It is defined in a neighborhood N of x_0. If $r \in R(S^k, S^l)$, then $rN = rNr^{-1} = N$ because of the invariance of ϕ. Therefore the left translation rN of N to the point $x = rx_0$ (r arbitrary) depends only on the left coset of $R(S^k, S^l)$ to which r belongs (i.e. on x alone). We denote this neighborhood of x by $N(x)$. These neighborhoods cover M_l^k.

For any pair x_1, x_2 in M_l^k such that $x_2 \in N(x_1)$, we choose an r_1 in R such that $r_1(X_1) = S^k$, and define

$$r(x_1, x_2) = r_1^{-1}\phi(r_1 x_2)r_1 .$$

It is easy to see that $r(x_1, x_2)$ is a rotation of S carrying X_1 into X_2. It appears to depend on the choice of r_1. This is not the case; for any other choice may be written $r_0 r_1$ for $r_0 \in R(S^k, S^l)$ and we have

$$(r_0 r_1)^{-1}\phi((r_0 r_1)x_2)(r_0 r_1) = r_1^{-1}\phi(r_1 x_2 r_0)r_1 = r_1^{-1}\phi(r_1 x_2)r_1 ,$$

because of the invariance of ϕ and the fact that x_2 is a left coset of $R(S^k, S^l)$. Finally $r(x_1, x_2)$ is continuous simultaneously in x_1, x_2. We may choose r_1 as a continuous function of x_1 neighboring a particular choice because R is a fibre bundle over M_l^k. If r_1 is replaced by this function in the definition of $r(x_1, x_2)$, the continuity of the latter becomes apparent.

Suppose now that g, g' are two maps of B in M_l^k such that

$$r(b) = r(g(b), g'(b))$$

is defined for every b. A map τ of $B \times S$ on itself is then defined by $\tau(b, y) = (b, y')$ where y' is the image of $y \in S$ under $r(b)$. It follows immediately that τ, restricted to $A(g)$, provides an equivalence between the bundles $A(g)$ and $A(g')$.

Finally, let $g(b, t)$, b in B, $0 \leqq t \leqq 1$, be a homotopy in M_l^k of the map $g(b) = g(b, 0)$. Since B is compact, a familiar uniformity argument shows that there exists a $\delta > 0$ such that $|t - t'| < \delta$ implies $g(b, t')$ is in $N(g, b, t))$ for any b in B. Introduce now a subdivision of the t-interval of mesh $< \delta$. As shown in the preceding paragraph, the bundles over B corresponding to successive subdivision points are equivalent. Since equivalence is a transitive relation, Theorem 3 is proved.

REMARK: If B is not compact, but the homotopy $g(b, t)$ is *uniform* in the sense described above, the conclusion will still hold.

12. Proof of Theorem 4

A simplex of B is denoted by σ, its closure by $\bar{\sigma}$. We shall suppose that the subdivision of B is so fine that each $\bar{\sigma}$ lies wholly within some coordinate neighborhood of the bundle A. We choose one such neighborhood for each σ and denote it by $N(\sigma)$. The corresponding coordinate function is denoted by ϕ_σ.

Our procedure will be to construct stepwise over the simplexes of B, in the order of their dimension, a system of functions λ_σ from which we shall be able to deduce the existence of a map g and an equivalence τ between A and $A(g)$.

The function λ_σ is to be a continuous map of $\bar{\sigma}$ in the space N_l^k of §9 so that $\lambda_\sigma(b)$ for each $b \in \bar{\sigma}$ is an orthogonal map of S^k into S. These functions are to have the following basic property:

(P) If σ' is a face of σ and b is in $\bar{\sigma}'$, then the orthogonal map $\phi_{\sigma' b}^{-1}\phi_{\sigma b}$ of S^k on itself followed by the map $\lambda_{\sigma'}(b)$ of S^k into S is the map $\lambda_\sigma(b)$ of S^k into S.

We define λ_σ arbitrarily over the vertices σ of B. Suppose, inductively, that λ_σ has been defined for each σ of dimension $<i$ so that (P) holds. For an i-simplex σ and a point b of its boundary, we define $\lambda_\sigma(b)$ by choosing a face σ' of σ such that $b \in \bar{\sigma}'$ and letting $\lambda_\sigma(b)$ be the orthogonal map $\phi_{\sigma' b}^{-1}\phi_{\sigma b}$ of S^k on itself followed by $\lambda_{\sigma'}(b)$. By (P), $\lambda_\sigma(b)$ is independent of the choice of $\bar{\sigma}' \supset b$. Since $\lambda_\sigma(b)$ is continuous over each closed face of σ, it is continuous over the boundary of σ. Since $l \geqq \dim B - 1$, we have $i - 1 \leqq l$. Therefore, by the lemma of §9, λ_σ may be extended continuously over σ. Let this be done for each i-simplex. By the construction, (P) holds. Thus the general step in the induction is complete.

If η denotes the natural map $N_l^k \to M_l^k$, we notice that, if $b \in \bar{\sigma}' \subset \bar{\sigma}$, then

$$\eta\lambda_\sigma(b) = \eta\lambda_{\sigma'}(b) = g(b),$$

for $\lambda_\sigma(b)$ and $\lambda_{\sigma'}(b)$ map S^k into the same great k-sphere of S. Since g is continuous over each closed $\bar{\sigma}$, it is continuous over B and defines thereby a bundle $A(g)$.

Denote the projection $A \to B$ by ψ. For any $a \in A$, we choose a σ such that $b = \psi(a) \in \bar{\sigma}$, and denote by $h(a)$ the image of a in S under the map $\phi_{\sigma b}$ followed by $\lambda_\sigma(b)$. By the property (P), $h(a)$ is independent of the choice of σ. It is a continuous function since it is continuous over each of the closed sets $\psi^{-1}(\sigma)$ which cover A. Since both $\phi_{\sigma b}$ and $\lambda_\sigma(b)$ are 1-1 maps, $h(a)$ is 1-1 for a restricted to $\psi^{-1}(b)$.

We define the map τ of A in $B \times S$ by

$$\tau(a) = (\psi(a), h(a)).$$

From the stated properties of h, it follows immediately that τ defines an equivalence between A and $A(g)$.

13. Proof of Theorem 5

Let g_0, g_1 be two maps of a complex B in M^k such that $A(g_0)$, $A(g_1)$ are equivalent bundles. Let I be the t-interval $0 \leqq t \leqq 1$, let $B' = B \times I$, and $B_t = B \times t$.

Define $g'(b, t) = g_0(b)$. In this way a bundle $A(g')$ over B' is obtained. If we identify B with B_0, the bundle $A(g_0)$ becomes identified with the part of $A(g')$ over B_0; because $g'(b, 0) = g_0(b)$.

Let $A(g_1')$ be the part of $A(g')$ over B_1. The map of $B_1 \times S$ onto $B_0 \times S$ defined by $((b, 1), y) \to ((b, 0), y)$ provides an equivalence between $A(g_1')$ and $A(g_0)$. Since the latter is equivalent to $A(g_1)$, the two equivalences provide an equivalence map τ of $A(g_1')$ onto $A(g_1)$.

Let $g(b, 0) = g_0(b)$, and $g(b, 1) = g_1(b)$. Our problem is to extend $g(b, t)$ over all of B'. To do this, we shall use the method of §12 with B' in place of B and $A(g')$ in place of A. In the present case, however, the function g is already specified on the subcomplex $B_0 + B_1$. Therefore, in order to apply §12, it is necessary to define first the functions λ_σ for σ in $B_0 + B_1$ so as to satisfy (P), and, in addition, $\eta\lambda_\sigma(b') = g(b')$ for each $b' \in \bar{\sigma}$. Once this is accomplished, the general step in the extension will follow as in §12 because $l \geqq \dim B \geqq \dim B' - 1$.

The essential point in obtaining the λ's for σ in $B_0 + B_1$ is that the function τ of §12 (defined there in terms of the λ's) is already given, in the present case, in the portion of $A(g')$ over $B_0 + B_1$. On $A(g_0)$, τ is the identity, on $A(g_1')$, it is the given equivalence with $A(g_1)$. We can use the τ to define the λ's.

As in §12, we suppose B' subdivided so fine that each closed cell lies wholly in a coordinate neighborhood for $A(g')$. We choose one such for each cell σ and denote the corresponding coordinate function by ϕ_σ. Map $B' \times S$ onto S by $\zeta(b', y) = y$. For σ in $B_0 + B_1$ and b' in $\bar{\sigma}$, define

$$\lambda_\sigma(b') = \zeta\tau\phi_{\sigma b'}.$$

In words, $\lambda_\sigma(b')$ is the orthogonal map of S^k into S obtained as the composition of the map $\phi_{\sigma b'}$ of S^k on the fibre over b' in $A(g')$, the equivalence map τ of this fibre into $B' \times S$, and finally the map ζ of this last image into S. One verifies immediately that (P) holds and $\eta\lambda_\sigma(b') = g(b')$. This completes the proof.

14. General extension theorem

Experience has shown that in most homotopy classification problems there is a basic extension theorem from which all results follow. We state here a single extension theorem which implies both Theorem 4 and Theorem 5.

THEOREM 12. *Let A be a k-sphere bundle over the complex B, let B' be a closed subcomplex of B, let g' be a map of B' in M_l^k, and let τ' be an equivalence map of the part of A over B' onto $A(g')$. If $l \geqq \dim B - 1$, then there exists an extension g of g' to all of B such that A and $A(g)$ are equivalent. Moreover this last equivalence may be chosen to be an extension of τ'.*

The proof can be found in sections 12 and 13.

THE UNIVERSITY OF MICHIGAN

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MULTIPLE INTEGRAL PROBLEMS IN PARAMETRIC FORM IN THE CALCULUS OF VARIATIONS[1]

BY J. ERNEST WILKINS, JR.

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1. Introduction

It is the purpose of this paper to give a discussion of the simplest problem for multiple integrals in parametric form of the calculus of variations, namely, the problem of minimizing a multiple integral

$$(1:1) \qquad I(V) = \int \cdots \int f(y^1, \cdots, y^n, p_1^1, \cdots, p_m^n)\, dt_1 \cdots dt_m,$$

where $p_a^i = \partial y^i / \partial t_a (i = 1, \cdots, n; a = 1, \cdots, m < n)$, in a class of varieties represented parametrically by equations of the form $y^i = y^i(t)$, all of which have the same boundary. In order that the integral (1:1) be independent of the parametric representation of the variety, it is known [XI] that $f(y, p)$ must satisfy the homogeneity condition

$$(1:2) \qquad f(y, pB) = |B| f(y, p),$$

where B is an arbitrary m-rowed square matrix with positive determinant $|B|$, and pB is the matrix product of the matrices p and B.

In Section 3 we develop some of the elementary consequences of the homogeneity condition (1:2). For the most part these consequences are direct generalizations of results known to be true for $m = 1$. In Section 5 we give additional consequences of the identity (1:2) which involve the derivatives of $f(y, p)$ of arbitrary order. In order to describe these results in a reasonably compact form, we find it desirable to employ certain notational conventions which we explain in Section 4. In case $n = m + 1$ certain additional consequences of the homogeneity condition are deduced in Section 6.

After formulating the problem of the calculus of variations in Section 7, we proceed to discuss the necessary conditions for a minimum. With the exception of the Jacobi condition, these necessary conditions are immediate consequences of the corresponding necessary conditions for non-parametric problems. For the conditions of Weierstrass and Legendre, no satisfactory result is known for non-parametric problems and consequently a similar situation exists for parametric problems. One apparent objection is that it is not true that the ordinary E-function $E_0(y, p, P)$ must be non-negative for all choices of the matrix $P \neq p$, but only if $P - p$ has rank one. For an example, see Bliss [I, p. 80]. Caratheodory [V] has proposed an E-function $E_c(y, p, P)$ which he found useful in

[1] This paper is a revision of a doctor's dissertation accepted by the University of Chicago. The author is indebted to Professor M. R. Hestenes for many helpful comments.

making sufficiency proofs. We give in Section 9 an example to show that $E_c \geqq 0$ is not necessary for a minimum. By analyzing the properties which an E-function should possess, we arrive at a new E-function $E_1(y, p, P)$ which possesses all of these properties and which reduces to E_0 in case $P - p$ has rank one. Moreover, this function vanishes identically for integrands of invariant integrals, although E_0 and E_c do not possess this property. On the basis of these results for the Weierstrass condition, we state the analogous results for the Legendre condition. We state and prove a Jacobi condition in Section 11.

Because of the hiati in the field theory for non-parametric problems, we are not able to obtain a complete field theory for our parametric problem. We can, however, define a field in Section 12 and state conditions necessary and sufficient that an integral (1:1) be invariant in Section 13.

2. Integrals independent of the parametric representation

Suppose that we are given an integral of the form

$$I(V) = \int_T f(t, y, p)\, dT, \tag{2:1}$$

where $t = (t_1, \cdots, t_m)$, $y = (y^1, \cdots, y^n)$, $p = (p_a^i) = (\partial y^i/\partial t_a)$ $(i = 1, \cdots, n; a = 1, \cdots, m < n)$, $dT = dt_1\, dt_2 \cdots dt_m$. The variety or manifold V will be represented by functions of the form $y^i = y^i(t)$ for t in some region T such that the matrix p has rank m on T. When these functions are of class C', i.e., when they have continuous first partial derivatives in T, the integral (2:1) is well-defined. For a parametric problem, however, we must require that the integral (2:1) have the same value for any parametric representation of the same variety V, regarded as being oriented in a fixed manner. The condition for this was investigated by Lewis [XI] and we state it as

THEOREM 2:1. *The integral* (2:1) *is independent of the parametric representation if and only if the integrand function is a function $f(y, p)$ independent of t and such that for every m-rowed square matrix B with positive determinant $|B|$ the homogeneity condition*

$$f(y, pB) = |B| f(y, p) \tag{2:2}$$

holds. In this equation pB is the matrix product of p and B.

The integrand function $f(y, p)$ can also be regarded as a function of y and the m-rowed minors of the matrix p as has been shown by Grosz [VIII]. This representation will be of little use to us unless $n = m + 1$ in which case his result may be stated as

THEOREM 2:2. *In case $n = m + 1$ define the m-rowed minors A_i of p to be the cofactors of x_i in the matrix $(p_a^i\, x_i)$. Then the integral* (2:1) *is independent of the parametric representation if and only if the integrand is a function $G(y, A)$ which is positively homogeneous of the first degree in the variables A_i.*

Since the matrix p has rank m it follows easily that there exists a matrix $q = (q_j^i)$ $(j = 1, \cdots, n - m)$ of rank $n - m$ such that

$$p'q = 0, \qquad |\, p'p \,| = |\, p\, q \,|, \tag{2:3}$$

where the prime denotes transposition. The matrix q is of course not uniquely determined by p since qC will satisfy (2:3) for any $(n - m)$-rowed square matrix C with determinant unity. Moreover, if q and q^* are determined by p and p^*, then because of the second condition in (2:3) $q^* = qC$ if and only if $p^* = pB$ with $|\, C \,| = |\, B \,|$. With these comments in mind it is easy to prove that the integrand of an integral (2:1) which is independent of the parametric representation may also be regarded as a function $H(y, q)$ which satisfies the homogeneity condition

$$H(y, qC) = |\, C \,| H(y, q) \tag{2:4}$$

for every $(n - m)$-rowed square matrix C with positive determinant. This criterion is not especially valuable to us unless $n = m + 1$, in which case it is equivalent to that supplied by Theorem 2:2. It does, however, display a type of duality in our problem in which $n - m$ and m are interchanged.

3. First consequences of the homogeneity condition

During the remainder of this paper we shall assume that $f(y, p)$ is a function satisfying the homogeneity condition (2:2) which possesses as many derivatives as are needed in any particular discussion. In this section certain elementary consequences of equation (2:2) will be deduced. Most of these results may be found in the paper of Lewis [XI] already referred to, but our proof of the fundamental relation (3:1) is considerably simpler.

THEOREM 3:1. *Let* f_i, f^a_i, f^a_{ik}, f^{ab}_{ik} $(a, b = 1, \cdots, m;\ i, k = 1, \cdots, n)$ *be the derivatives*

$$f_i = \partial f/\partial y^i, \qquad f^a_i = \partial f/\partial p^i_a, \qquad f^a_{ik} = \partial f_k/\partial p^i_a, \qquad f^{ab}_{ik} = \partial f^a_i/\partial p^k_b.$$

Then, with the notation C^b_a *for the cofactor of* B^a_b *in the matrix* B, *the following identities are satisfied:*[2]

$$p^i_a f^b_i = \delta^b_a f; \tag{3:1}$$

$$f_i(y, pB) = |\, B \,| f_i(y, p); \tag{3:2}$$

$$p^i_a f^b_{ik} = \delta^b_a f_k; \tag{3:3}$$

$$B^a_b f^b_i(y, pB) = |\, B \,| f^a_i(y, p); \tag{3:4}$$

$$f^b_i(y, pB) = C^b_a f^a_i(y, p); \tag{3:5}$$

$$p^i_a f^{bc}_{ik} = \delta^b_a f^c_k - \delta^c_a f^b_k = (\delta^b_a \delta^c_d - \delta^c_a \delta^b_d) f^d_k; \tag{3:6}$$

$$p^i_a (f^{bc}_{ik} + f^{cb}_{ik}) = 0; \tag{3:7}$$

$$p^i_a p^k_a f^{bc}_{ik} = (\delta^b_a \delta^c_d - \delta^c_a \delta^b_d) f. \tag{3:8}$$

[2] We employ the usual convention concerning summation over repeated indices.

In all of these formulas the symbol δ_a^b is the Kronecker delta which has the value one for $b = a$ and the value zero otherwise. We obtain (3:1) by differentiating (2:2) with respect to B_b^a and setting B equal to the identity matrix. The other identities are easily obtained from (2:2) and this one by appropriate differentiations.

THEOREM 3:2. *There exist functions* R_{jl}^{ab}, C_{jl}^{ab} $(a, b = 1, \cdots, m; j, l = 1, \cdots, n - m)$ *such that*

$$R_{jl}^{ab} = R_{jl}^{ba} = R_{lj}^{ab}, \quad C_{jl}^{ab} = -C_{jl}^{ba} = -C_{lj}^{ab}, \tag{3:9}$$

$$f_{ik}^{ab} + f_{ik}^{ba} = 2R_{jl}^{ab} q_j^i q_l^k, \tag{3:10}$$

$$f(f_{ik}^{ab} - f_{ik}^{ba}) - 2(f_i^a f_k^b - f_i^b f_k^a) = 2C_{jl}^{ab} q_j^i q_l^k, \tag{3:11}$$

where the q_j^i *were defined in Section 2.*

By (3:7) there exist quantities A_{kj}^{ab} such that $f_{ik}^{ab} + f_{ik}^{ba} = A_{kj}^{ab} q_j^i = A_{ij}^{ab} q_j^k$. Thus $(A_{ij}^{ab} p_c^i) q_j^k = 0$, $A_{ij}^{ab} p_c^i = 0$, so that there exist quantities R_{jl}^{ab} such that $A_{ij}^{ab} = 2R_{jl}^{ab} q_l^i$, and then (3:10) holds. The symmetric relations expressed by the first equations in (3:9) are then easy consequences of the fact that the matrix q has rank $n - m$. To prove the second part of the theorem, we find by using (3:1) and (3:6) that

$$p_c^i[f(f_{ik}^{ab} - f_{ik}^{ba}) - 2(f_i^a f_k^b - f_i^b f_k^a)] = 0.$$

Then an argument precisely the same as that used in proving the first part of the theorem suffices to complete the proof.

The Euler expressions $T_i(V)$ for a variety V admitting a representation $y^i = y^i(t)$ of class C'' are defined as

$$T_i(V) \equiv f_i - \partial f_i^b / \partial t_b, \tag{3:12}$$

where the arguments of the derivatives of f are those belonging to V. Let V be a fixed variety and define

$$2\omega(t, \eta, \pi) \equiv f_{ik}\eta^i\eta^k + 2f_{ki}^c \eta^i \pi_c^k + f_{ik}^{bc} \pi_b^i \pi_c^k \qquad (f_{ik} = \partial f_i / \partial y_k). \tag{3:13}$$

Then the Jacobi expressions $J_i(\eta)$ are defined by

$$J_i(\eta) \equiv \omega_i - \partial \omega_i^b / \partial t_b, \tag{3:14}$$

where $\omega_i = \partial\omega/\partial\eta^i$, $\omega_i^b = \partial\omega/\partial\pi_b^i$, $\pi_b^i = \partial\eta^i/\partial t_b$.

THEOREM 3:3. *If* V *is a variety of class* C'', *the following identities subsist between its Euler and Jacobi expressions*:

$$T_i(V) p_a^i = 0 \tag{3:15}$$

$$J_i(\eta) p_a^i = -T_i(V) \pi_a^i. \tag{3:16}$$

Moreover, if V *is of class* C''', *then for every set of functions* $Z^a(t)$ *of class* C'' *the functions* $\phi^i = p_a^i Z^a$ *are of class* C'' *and we have*

$$J_i(\phi) = \partial(Z^a T_i)/\partial t_a. \tag{3:17}$$

Using (3:1) equation (3:15) follows from the relation

$$\begin{aligned} T_i(V)p_a^i &= f_i p_a^i - \partial(p_a^i f_i^b)/\partial t_b + p_{ab}^i f_i^b \\ &= \partial f/\partial t_a - \partial(\delta_a^b f)/\partial t_b = 0, \end{aligned}$$

where $p_{ab}^i = \partial p_a^i/\partial t_b = p_{ba}^i$. Replacing in (3:15) the functions $y^i(t)$ defining V by $y^i(t) + e\eta^i(t)$, differentiating with respect to e and setting $e = 0$ gives (3:16). Letting $Z_b^a = \partial Z^a/\partial t_b$, we find with the help of (3:3) and (3:6) that

$$\begin{aligned} \omega_i(\phi) &= (f_{ik}p_a^k + f_{ki}^c p_{ac}^k)Z^a + (f_{ki}^c p_a^k)Z_c^a \\ &= (\partial f_i/\partial t_a)Z^a + (\delta_a^c f_i)Z_c^a = \partial(f_i Z^a)/\partial t_a, \\ \omega_i^b(\phi) &= (f_{ik}^b p_a^k + f_{ik}^{bc} p_{ac}^k)Z^a + (f_{ik}^{bc} p_a^k)Z_c^a \\ &= (\partial f_i^b/\partial t_a)Z^a + (\delta_a^c f_i^b - \delta_a^b f_i^c)Z_c^a \\ &= \partial(f_i^b Z^a)/\partial t_a - f_i^c Z_c^b. \end{aligned}$$

Then a little manipulation shows that

$$\partial\omega_i^b(\phi)/\partial t_b = \partial(Z^a \partial f_i^b/\partial t_b)/\partial t_a.$$

Combining this equation with the last equation for $\omega_i(\phi)$ gives (3:17).

4. Notations to be employed in subsequent sections

For the sake of convenience of reference we list here the principal notations which we shall use later. We shall use the letters a, b, c, d, e, r and s, with or without subscripts as indices having the range 1 to m, while the letters i and k will be indices with the range 1 to n, and the letters j and l will be indices with the range 1 to $n - m$. Ordered collections of the letters a, b, c, d, e, i, k, r and s will be denoted by the corresponding Greek letter. Thus

$$\begin{aligned} &\alpha = (a_1, \cdots, a_g), \quad \beta = (b_1, \cdots, b_g), \quad \gamma = (c_1, \cdots, c_h), \\ &\delta = (d_1, \cdots, d_h), \quad \epsilon = (e_1, \cdots, e_h), \quad \iota = (i_1, \cdots, i_g), \\ &\kappa = (k_1, \cdots, k_h), \quad \rho = (r_1, \cdots, r_g), \quad \sigma = (s_1, \cdots, s_h). \end{aligned}$$

Unless otherwise specified, the number of components in such a collection will always be g or h as given above.

If α and ι are given as above, then we shall use the symbol p_α^ι to stand for the product $p_{a_1}^{i_1} p_{a_2}^{i_2} \cdots p_{a_g}^{i_g}$, and the symbol f_ι^α to stand for the repeated partial derivative of f with respect to $p_{a_1}^{i_1}, \cdots, p_{a_g}^{i_g}$. By a summation such as $p_\alpha^\iota f_\iota^\beta$ we mean a summation over $i_1, \cdots, i_g$ as each of these indices ranges independently from 1 to n.

For any m-rowed square matrix $B = (B_b^a)$ we shall use the symbol B_β^α to stand for the product $B_{b_1}^{a_1} \cdots B_{b_g}^{a_g}$. In particular we shall use this notation for the identity matrix, so that $\delta_\beta^\alpha = 1$ in case $\alpha = \beta$ and $\delta_\beta^\alpha = 0$ otherwise.

Let α and β be as above and define D^α_β to be the g-rowed minor of the identity matrix whose rows are those given in the superscript α and whose columns are those given in the subscript β. Then it is clear that $D^\alpha_\beta = 1$ in case the $b_1, \cdots, b_g$ are all distinct and the components of α form an even permutation of those of β, $D^\alpha_\beta = -1$ in case the b's are distinct and the a's are an odd permutation of the b's, while $D^\alpha_\beta = 0$ otherwise. This quantity D^α_β is the generalized Kronecker symbol discussed in detail by Murnaghan [XIV, XV]. It may also be defined as the result obtained when the determinant $|B| = |B^a_b|$ is differentiated with respect to $B^{a_1}_{b_1}, \cdots, B^{a_g}_{b_g}$ in turn and then B^a_b is set equal to δ^a_b. It is clear that if αQ denotes the collection resulting when a permutation Q is applied to the collection α and βR is a similar permutation of β, we have

$$D^{\alpha Q}_{\beta R} = (-1)^Q(-1)^R D^\alpha_\beta, \tag{4:1}$$

$$D^\alpha_\beta = \sum(-1)^Q\delta^{\alpha Q}_\beta = \sum(-1)^Q\delta^\alpha_{\beta Q}, \tag{4:2}$$

where $(-1)^Q = 1$ or -1 according as Q is an even or an odd permutation and the sums are taken for all permutations Q on g letters. We remark with Murnaghan that the following identities hold, in which $B = (B^a_b)$ is now an h-rowed square matrix:

$$D^\gamma_\delta = D^\delta_\gamma = D^{\gamma\epsilon}_{\epsilon\delta} = (D^\gamma_\epsilon D^\epsilon_\delta)/h!, \tag{4:3}$$

$$|B|\,D^\gamma_\delta = D^\epsilon_\delta B^\gamma_\epsilon = D^{\gamma\epsilon}_{\epsilon\delta}. \tag{4:4}$$

In the next section we shall have occasion to use a set of quantities $M^{\beta;\epsilon}_{a;\gamma}$. These constants will be defined explicitly in Section 15. For the present it is sufficient to define them inductively as follows:

$$M^{b;e}_{a;c} = D^{be}_{ac}, \tag{4:5}$$

$$M^{b;ee}_{a;\gamma e} = M^{b;e}_{a;\gamma}\delta^e_e - \delta^{ebe}_{\gamma ea}, \tag{4:6}$$

$$M^{\beta b;\epsilon}_{\alpha a;\gamma} = M^{\beta;b\epsilon}_{\alpha;d\delta}M^{d;\delta}_{a;\gamma}. \tag{4:7}$$

5. Further consequences of the homogeneity condition

In this section we shall investigate the implications of the homogeneity condition (2:2) for the higher derivatives of f. In particular formulas generalizing (3:1), (3:5) and (3:6) will be obtained. The formula generalizing (3:1) is that stated in the following

THEOREM 5:1. *If $f(y, p)$ satisfies (2:2), then*

$$p^i_\alpha f^\beta_i = D^\alpha_\beta f. \tag{5:1}$$

This follows at once upon differentiating (2:2) with respect to $B^{a_1}_{b_1}, \cdots, B^{a_g}_{b_g}$ and setting $B^a_b = \delta^a_b$.

The formula generalizing (3:5) is stated in

THEOREM 5:2. *If $f(y, p)$ satisfies (2:2), then for each m-rowed square matrix B with positive determinant we have*

$$(5:2) \qquad |B|^{g-1} f_i^\beta(y, pB) = C_\alpha^\beta f_i^\alpha(y, p),$$

where C_a^b is the cofactor of B_b^a in the matrix B. Hence we also have, when $g = m$,

$$(5:3) \qquad D_\beta^\alpha f_i^\beta(y, pB) = D_\beta^\alpha f_i^\beta(y, p).$$

If $g = 1$, we see from (3:5) that (5:2) is true. Assume that (5:2) is true for a fixed g and differentiate it with respect to p_a^i. We get

$$|B|^{g-1} B_c^a f_{ij}^{\beta c}(y, pB) = C_\alpha^\beta f_{ij}^{\alpha a}(y, p).$$

If we multiply by C_a^b and sum for a, we find that

$$|B|^g f_{ij}^{\beta b}(y, pB) = C_{\alpha a}^{\beta b} f_{ij}^{\alpha a}(y, p).$$

This completes the induction and proves (5:2). Multiplication of (5:2) for $g = m$ by D_β^γ with $\gamma = (c_1, \cdots, c_m)$ yields, when we apply (4:4) to the matrix C,

$$|B|^{m-1} D_\beta^\gamma f_i^\beta(y, pB) = D_\alpha^\gamma |C| f_i^\alpha(y, p),$$

whence (5:3) follows, since $|C| = |B|^{m-1} > 0$.

The generalization of (5:6) is stated in

THEOREM 5:3. *If $f(y, p)$ satisfies (2:2), then*

$$(5:4) \qquad p_\alpha^i f_{ik}^{\beta\epsilon} = M_{\alpha;\gamma}^{\beta;\epsilon} f_k^\gamma,$$

where $M_{\alpha;\gamma}^{\beta;\epsilon}$ is defined by equations (4:5), (4:6), (4:7).

For $g = h = 1$ the equation (5:4) becomes, when we use (4:5), precisely (3:6) and is therefore true. Assume that (5:4) is true for $g = 1$ and a fixed h. Differentiating with respect to p_e^k gives

$$p_\alpha^i f_{ikk}^{b\epsilon e} + \delta_\alpha^e f_{ik}^{b\epsilon} = M_{\alpha;\gamma}^{b;\epsilon} f_{kk}^{\gamma e},$$

$$p_\alpha^i f_{ikk}^{b\epsilon e} = (M_{\alpha;\gamma}^{b;\epsilon} \delta_c^e - \delta_{\gamma c\alpha}^{\epsilon b e}) f_{kk}^{\gamma c} = M_{\alpha;\gamma c}^{b;\epsilon e} f_{kk}^{\gamma c},$$

by virtue of (4:6). Hence (5:4) is true for $g = 1$ and all h. Assume it true for a fixed g and all h. Then we have

$$p_{\alpha a}^{ij} f_{ijk}^{\beta b\epsilon} = M_{\alpha;d\delta}^{\beta;b\epsilon} (p_\alpha^i f_{ik}^{d\delta}) = M_{\alpha;d\delta}^{\beta;b\epsilon} M_{a;\gamma}^{d;\delta} f_k^\gamma = M_{\alpha a;\gamma}^{\beta b;\epsilon} f_k^\gamma,$$

when we use (5:4) for $g = 1$ and (4:7). This completes the induction and proves (5:4) for all values of g and h.

6. The case when $n = m + 1$

In this case certain additional consequences of the homogeneity condition are obtainable, and we shall find these results useful in comparing the work of various authors. By Theorem 2:2 we have $f(y, p) = G(y, A)$, where G satisfies

$$(6:1) \qquad G(y, KA) = K\, G(y, A)$$

for every positive number K. Then it follows easily that we have

$$A_iG_i = G \qquad (G_i = \partial G/\partial A_i), \tag{6:2}$$

$$A_iG_{ik} = 0 \qquad (G_{ik} = \partial G_i/\partial A_k = G_{ki}). \tag{6:3}$$

We remark with Radon [XVIII] that (6:3) implies the existence of quantities $\theta^{ab} = \theta^{ba}$ such that

$$G_{ik} = p_a^i \theta^{ab} p_b^k . \tag{6:4}$$

In order to derive a formula for the derivatives G_i in terms of the derivatives f_i^a we remark that the definition of A_i as the cofactor of x_i in the matrix $(p_a^i\, x_i)$ enables us to write (6:2) as

$$f(y, p) = G(y, A) = |\, p_a^i\, G_i\,|. \tag{6:5}$$

Then $f_i^a = G_k \partial A_k/\partial p_a^i$ is the cofactor of p_a^i in the above determinant. It follows that $f_i^a G_i = 0$. We also have the equations $p_a^i A_i = 0$, expressing the perpendicularity of the normal vector to the tangent vectors. Using (3:1) and the relations derived above we readily find that $|\, p_a^i\, G_i\,|\,|\, f_i^a\, A_i\,| = f^{m+1}$. If $f \neq 0$, we therefore have $A_iF_i = f^m$, where F_i is the cofactor of A_i in the determinant $|\, f_i^a\, A_i\,|$. With this definition of F_i we see that $f_i^a F_i = 0$. Consequently we have

$$A_i(F_i - G_i f^{m-1}) = 0, \qquad f_i^a(F_i - G_i f^{m-1}) = 0.$$

Since $|\, f_i^a\, A_i\,| = f^m \neq 0$, we conclude that

$$F_i = G_i f^{m-1}, \tag{6:6}$$

at least at all points where $f \neq 0$. If $m = 1$, the relation (6:6) is trivially established in the form $F_i = G_i$, regardless of the vanishing of f. If $m > 1$ and $f = 0$, then equations (3:1) show that the matrix (f_i^a) has rank at most 1, so that $F_i = 0$. In this case both sides of (6:6) are zero, so that the equation (6:6) holds without restriction. Thus we have proved the first part of

THEOREM 6:1. *If $n = m + 1$, so that $f(y, p) = G(y, A)$, then the first derivatives of G with respect to the variables A_i satisfy* (6:6) *when the F_i are the cofactors of A_i in the matrix $(f_i^a\, A_i)$. If we define a symbol f_β^ι by the equation*

$$f_\beta^\iota = f_{i_1}^{b_1} f_{i_2}^{b_2} \cdots f_{i_g}^{b_g}, \tag{6:7}$$

we have, at each point where $f \neq 0$,

$$G_i = (-1)^{n+i} f^{1-m} D_{\alpha_0}^{\alpha} f_{\alpha}^{\kappa_i}, \qquad (i \text{ not summed}), \tag{6:8}$$

where $\alpha_0 = (1, 2, \cdots, m)$, $\kappa_i = (1, \cdots, i-1, i+1, \cdots, m)$ and $\alpha = (a_1, \cdots, a_m)$. The symbol f_β^ι is related to the symbol f_ι^β by the identity

$$f^{h-1} D_\epsilon^\gamma f_\kappa^\epsilon = h! D_\epsilon^\gamma f_\iota^\kappa, \tag{6:9}$$

where γ, ϵ and κ are defined in Section 4. Therefore we also have

$$G_i = (-1)^{n+i} (D_{\alpha_0}^{\alpha} f_{\kappa_i}^{\alpha})/m! \qquad (i \text{ not summed}). \tag{6:10}$$

We get (6:8) by expanding the cofactor F_i according to the rule given in (4:4). We observe that (6:9) is trivially true for $h = 1$. By (5: 4) and (3:1) we have

$$p_d^{k_1} f_\kappa^\epsilon = M_{d;\delta}^{\epsilon_1;\epsilon'} f_{\kappa'}^\delta, \qquad p_d^{k_1} f_\epsilon^\kappa = \delta_d^{\epsilon_1} f f_\epsilon^{\kappa'}, \tag{6:11}$$

where the sum is for the k_1 occurring explicitly in $p_d^{k_1}$ and implicitly in the subscript κ. Here $\kappa' = (k_2, \cdots, k_h)$, $\epsilon' = (e_2, \cdots, e_h)$ and $\delta = (d_1, \cdots, d_{h-1})$. It will be shown in Section 15 that

$$D_{\rho\sigma}^{\beta\epsilon} M_{\alpha;\gamma}^{\rho;\sigma} = \frac{(g+h)!}{h!} D_{\alpha\gamma}^{\beta\epsilon}. \tag{6:12}$$

Using this formula and equation (6:11) we find that

$$p_d^{k_1}(f^{h-1} D_\epsilon^\gamma f_\kappa^\epsilon - h!\, D_\epsilon^\gamma f_\epsilon^\kappa) = hf(f^{h-2} D_{d\delta}^\gamma f_{\kappa'}^\delta - (h-1)!\, D_{d\delta}^\gamma f_\delta^{\kappa'}).$$

Assume now that (6:9) is true for $h - 1$. Since we have from the definition of a determinant that

$$D_{d\delta}^\gamma = \sum_{\lambda=1}^{h} (-1)^{\lambda-1} \delta_d^{c_\lambda} D_\delta^{\gamma\lambda},$$

where $\gamma_\lambda = (c_1, \cdots, c_{\lambda-1}, c_{\lambda+1}, \cdots, c_h)$, and since by the hypothesis of the induction we have

$$f^{h-2} D_\delta^{\gamma\lambda} f_{\kappa'}^\delta - (h-1)!\, D_\delta^{\gamma\lambda} f_\delta^{\kappa'} = 0,$$

we conclude that

$$p_d^{k_1}(f^{h-1} D_\epsilon^\gamma f_\kappa^\epsilon - h!\, D_\epsilon^\gamma f_\epsilon^\kappa) = 0.$$

Since the quantity in the parenthesis is changed in sign by an interchange of two of the components of κ, we conclude that there exist quantities C^γ such that

$$f^{h-1} D_\epsilon^\gamma f_\kappa^\epsilon - h!\, D_\epsilon^\gamma f_\epsilon^\kappa = C^\gamma A_{k_1} A_{k_2} \cdots A_{k_h}.$$

The right hand side of this equation is unaltered by an interchange of two of the components of κ while the left hand side is changed in sign. Hence both sides are zero. This completes the induction and proves (6:9). Then (6:9) for $h = m$ and (6:8) imply (6:10).

That the formula (6:9) does not hold in general if $n \neq m + 1$ (or $m \neq 1$) is shown by the following example with $m = 2$, $n = 4$. Let

$$f(y, p) = [(p_1^1 p_2^2 - p_1^2 p_2^1)(p_1^3 p_2^4 - p_1^4 p_2^3)]^{\frac{1}{2}}.$$

Then $f(y, p)$ satisfies (2:2) and by actual computation we find that

$$f(f_{12}^{12} - f_{12}^{21}) - 2(f_1^1 f_2^2 - f_1^2 f_2^1) = \tfrac{1}{4}(p_1^3 p_2^4 - p_1^4 p_2^3) \neq 0.$$

This example also shows that the functions C_{jl}^{ab} of Theorem 3:2 are not identically zero in f, although this is true for $n = m + 1$ and for $m = 1$.

It is clear that a matrix q satisfying (2:3) will have one column and that we may take $q_1^i = A_i$. By (3:10) we have

$$f_{ik}^{ab} + f_{ik}^{ba} = 2R_{11}^{ab}A_iA_k. \tag{6:13}$$

THEOREM 6:2. *The second derivatives of G with respect to the variables A_i are given by* (6:4) *with* $\theta^{ab} = R_{11}^{ab}$.

Since $n = m + 1$, we find that

$$p_c^\mu \partial A_\mu/\partial p_a^i = -\delta_a^c A_i \qquad (\mu, \nu = 1, \cdots, n),$$

$$\partial^2 A_\mu/\partial p_a^i \partial p_b^k + \partial^2 A_\mu/\partial p_b^i \partial p_a^k = 0.$$

It follows that

$$\begin{aligned} f_{ik}^{ab} + f_{ik}^{ba} &= G_{\mu\nu}[(\partial A_\mu/\partial p_a^i)(\partial A_\nu/\partial p_b^k) + (\partial A_\mu/\partial p_b^i)(\partial A_\nu/\partial p_a^k)] \\ &= \theta^{cd}p_c^\mu p_d^\nu[(\partial A_\mu/\partial p_a^i)(\partial A_\nu/\partial p_b^k) + (\partial A_\mu/\partial p_b^i)(\partial A_\nu/\partial p_a^k)] \\ &= 2\theta^{ab}A_iA_k. \end{aligned}$$

Comparison of this last equation with (6:13) completes the proof of the theorem.

7. Analytic formulation of the problem

Because of the results established in Section 2, we may state our parametric multiple integral problem in the calculus of variations in the following manner. Let $\mathfrak{R}$ be a region in $(n + nm)$-dimensional space of points (y, p) with the property that if (y, p) is in $\mathfrak{R}$, then p has rank m and the point (y, pB) is also in $\mathfrak{R}$ for every m-rowed square matrix B with positive determinant. Let $f(y, p)$ be a function of class C^m on $\mathfrak{R}$ which satisfies the homogeneity condition (2:2).

By an admissible region T, we mean one in m-dimensional t-space whose boundary B can be subdivided into a finite number of non-singular $(m - 1)$-dimensional subsets of class C' whose intersections form a set E with the property that there exist positive numbers q and s with $s < m - 1$ such that, given a number $r > 0$, the quantity q/r^s is an upper bound of the number of points in a set E^* contained in E and chosen so that the distance between any two points of E^* is at least r. Such regions were considered by Carson [VI] and he showed that Green's lemma is valid for them.

By an admissible set of functions $y^i(t)$, we mean a set of functions of class C' on $T + B$, where T is an admissible region, for which the point $[y(t), p(t)]$ is in $\mathfrak{R}$ for t in $T + B$. We will say that two admissible sets of functions, $y^i(t)$ on $T + B$ and $Y^i(u)$ on $U + C$, are equivalent in case there exists a 1–1 transformation $t_a = t_a(u)$ of class C' between $T + B$ and $U + C$ for which $|\partial t_a/\partial u_b|$ is positive and such that $y^i(t(u)) = Y^i(u)$ on $U + C$. This relation is clearly reflexive, symmetric and transitive, and may therefore be used to classify admissible sets of functions into maximal classes of equivalent sets. We now define an admissible variety to be such a class of equivalent admissible sets of functions. The results of Section 2 show that the integral (1:1) defines a number which is the

same for all elements of a class of equivalent admissible sets of functions. Hence it depends only on the variety and we may set

$$I(V) = \int_T f(y, p)\, dT. \tag{7:1}$$

The calculus of variations problem to be considered may now be stated as the problem of finding that admissible variety V which gives the integral (7:1) its minimum value in the class of admissible varieties having the same boundary.

8. Necessary conditions associated with the first variation

Let V be an admissible variety represented by a set of functions $y^i(t)$ for t in $T + B$ and let L be its boundary. We say that the functions $\eta^i(t)$ define an admissible variation in case they are of class C' on $T + B$. The first variation of $I(V)$ along V then has the form

$$I_1(\eta) \equiv \int_T (f_i\, \eta^i + f_i^a\, \pi_a^i)\, dT \equiv \int_T \Phi(\eta)\, dT, \tag{8:1}$$

where $\pi_a^i = \partial \eta^i / \partial t_a$. If V is a minimizing variety, it is clear that $I_1(\eta) = 0$ for all admissible variations $\eta^i(t)$ such that $\eta^i(t) \equiv 0$ on B. Carson [VI] has shown that this will occur if and only if the Haar-Coral equations

$$\int_{T_0} f_i\, dT = \int_{B_0} f_i^a\, m_a\, dB \tag{8:2}$$

hold for every cube T_0 which together with its boundary B_0 is contained in T, where the m_a are the direction cosines of the outer normal to B_0. If moreover the functions $y^i(t)$ are of class C'', then the equations (8:2) hold if and only if the Euler-Lagrange equations

$$T_i \equiv f_i - \partial f_i^a / \partial t_a = 0 \tag{8:3}$$

are satisfied at each point of T. Solutions of these equations are called extremals. The equations (8:3) are not all independent, since (3:15) holds. But we see from (3:15) that there exist functions S_j $(j = 1, \cdots, n - m)$ such that $T_i = S_j q_j^i$, so that a variety is an extremal if and only if $S_j = 0$.

Suppose now that V is of class C''. Then there exist functions $u_j^i(t)$ of class C' on T such that the matrix (p, u) is non-singular. In fact, we could take $u = q$. For any admissible variations $\eta^i(t)$ the equations

$$\eta^i = p_a^i Z^a + u_j^i w^j \tag{8:4}$$

can be solved to obtain unique functions $Z^a(t)$ and $w^j(t)$ of class C' on T. Moreover, $\eta^i(t) \equiv 0$ on B if and only if $Z^a(t) \equiv w^j(t) \equiv 0$ on B. By straightforward calculation we find that $\Phi(\eta) = \Phi(uw) + \partial(Z^a f)/\partial t_a$. Then if $\eta^i(t) \equiv 0$ on B we see from Green's lemma that

$$I_1(\eta) = \int_T (M_j w^j + N_j^b w_b^j)\, dT \qquad (w_b^j = \partial w^j / \partial t_b), \tag{8:5}$$

where M_j and N_j^b are defined by

$$M_j = f_i u_j^i + f_i^b u_{jb}^i, \qquad N_j^b = f_i^b u_j^i \qquad (u_{jb}^i = \partial u_j^i/\partial t_b). \tag{8:6}$$

THEOREM 8:1. *Along an admissible variety V of class C'' the first variation is expressible in the form* (8:5), *where* $w = [(q'u)^{-1}q]\eta$. *The vanishing of the integral* (8:5) *for all functions* $w^j(t)$ *of class* C' *such that* $w^j(t) \equiv 0$ *on* B *is equivalent to the vanishing of the integral* (8:1) *for all admissible variations* $\eta^i(t)$ *such that* $\eta^i(t) \equiv 0$ *on* B.

We need merely verify the formula for w. The equation (8:4) can be written in matrix form as $pZ + uw = \eta$, whence multiplication by q', remembering that $q'p = 0$, shows that $q'uw = q'\eta$. The matrix $q'u$ is non-singular since both q and u have rank $n - m$, and consequently our formula for w is true.

The Euler-Lagrange equations for the integral (8:5) are

$$f_i u_j^i + f_i^b u_{jb}^i - \partial(f_i^a u_j^i)/\partial t_a = 0,$$

and these reduce immediately to $T_i u_j^i = S_l(q_l^i u_j^i) = 0$, equations which are equivalent to $S_j = 0$ since the matrix $q'u$ has rank $n - m$.

Ridge conditions could be stated for our problem. Since these conditions are precisely the same as for the non-parametric multiple integral problem treated by Powell [XVI] or Carson [VI], we shall not formulate them precisely here.

9. The conditions of Weierstrass and Legendre

The direct generalization of the necessary condition of Weierstrass for simpler problems would be to assert that the function

$$E_0(y, p, P) = f(y, P) - f(y, p) - (P_a^i - p_a^i)f_i^a(y, p) \tag{9:1}$$

is non-negative for every matrix P and for all points (y, p) belonging to the minimizing variety V. But a simple example (see Bliss [I, p. 80]) can be given to show that this condition is in general false, although McShane [XII], Weyl [XIX] and Graves [VII] have each proved that E_0 is non-negative whenever the matrix $P - p$ has rank one. Caratheodory [V] and Boerner [III, IV] have considered an E-function having the form

$$E_c(y, p, P) = f(y, P) - f^{1-m} \mid \delta_b^a f + (P_a^i - p_a^i)f_i^b \mid, \tag{9:2}$$

assuming for their theory that $f > 0$. By expanding the determinant as a polynomial in f we see that $E_c = E_0$ whenever $P - p$ has rank one. They found their function useful in making sufficiency proofs, but they did not consider the necessity of $E_c \geqq 0$. We shall give later on an example similar to that already quoted and show that $E_c \geqq 0$ is not necessary for a minimum. For our parametric problem E_c reduces to the expression

$$E_c(y, p, P) = f(y, P) - f^{1-m} \mid P_a^i f_i^b \mid, \tag{9:3}$$

since (3:1) holds. Computing the determinant we find that

$$E_c(y, p, P) = f(y, P) - f^{1-m}D^{\beta}_{\gamma}P^{\iota}_{\beta}f^{\gamma}_{\iota}/m!, \tag{9:4}$$

in which ι, β and γ are m-partite and f^{γ}_{ι} is the quantity defined in (6:7).

For the purpose of making sufficiency proofs we must have $E(y, p, P) = f(y, P) - H(y, p, P)$, where $H(y, P(y), p)$ has the form of the integrand of an invariant integral. It will be shown in Section 13 that H is necessarily of the form $H(y, p, P) = P^{\iota}_{\beta}C^{\beta}_{\iota}(y, p)$, where ι and β are m-partite and C^{β}_{ι} is skew in ι and β, i.e., an interchange of two of the components of either ι or β changes the sign of C^{β}_{ι}. Hence we must have

$$E(y, p, P) = f(y, P) - P^{\iota}_{\beta}C^{\beta}_{\iota}(y, p). \tag{9:5}$$

It then follows that we have

$$E(y, p, PB) = |B| E(y, p, P) \tag{9:6}$$

for every m-rowed square matrix B with a positive determinant. But we also require that E satisfy

$$E(y, pB, P) = E(y, p, P), \qquad E(y, p, p) = 0. \tag{9:7}$$

It is clear that the first equation in (9:7) is equivalent to

$$C^{\beta}_{\iota}(y, pB) = C^{\beta}_{\iota}(y, p). \tag{9:8}$$

Moreover, by virtue of (9:6), the second equation in (9:7) holds if and only if $E(y, p, pB) = 0$. Differentiation of this relation with respect to $B^{b_1}_{a_1}, \cdots, B^{b_m}_{a_m}$ gives, when we use (5:1),

$$p^{\iota}_{\beta}f^{\alpha}_{\iota} = m!\, p^{\iota}_{\beta}C^{\alpha}_{\iota}.$$

Defining a new quantity F^{γ}_{ι} by the equation

$$F^{\gamma}_{\iota} = D^{\gamma}_{\alpha}f^{\alpha}_{\iota}/m!, \tag{9:9}$$

we see from (5:1) that $p^{\iota}_{\beta}f^{\gamma}_{\iota} = p^{\iota}_{\beta}F^{\gamma}_{\iota}$. Hence the second equation in (9:7) holds if and only if

$$p^{\iota}_{\beta}F^{\gamma}_{\iota} = m!\, p^{\iota}_{\beta}C^{\gamma}_{\iota}. \tag{9:10}$$

The equations (9:8) and (9:10) do not suffice to determine the form of the E-function uniquely. For it is obvious that $C^{\gamma}_{\iota} = F^{\gamma}_{\iota}/m!$ satisfies (9:10) and it also satisfies (9:8) by virtue of (5:3). This choice of C^{γ}_{ι} gives an E-function of the form

$$E_1(y, p, P) = f(y, P) - P^{\iota}_{\beta}D^{\beta}_{\gamma}f^{\gamma}_{\iota}/(m!)^2. \tag{9:11}$$

But the function $E_c(y, p, P)$ given by (9:4) also has functions $C^{\beta}_{\iota} = f^{1-m}D^{\beta}_{\gamma}f^{\gamma}_{\iota}/m!$ which are easily seen to satisfy (9:8) and (9:10). For $n = m + 1$ we see from the definitions (9:4) and (9:11) and the identity (6:9) that $E_c = E_1$. The remarks following the proof of Theorem 6:1 show that E_c is not in general equal to E_1.

For the case $n = m + 1$, Radon [XVIII] has defined an E-function in terms of the function $G(y, A)$. This function is

$$E_r(y, A, A^*) = G(y, A^*) - A_h^* G_h(y, A) \quad (h = 1, \cdots, n), \tag{9:12}$$

and it should be remarked that McShane [XIII] has proved that $E_r \geqq 0$ is necessary for a minimum. If A_i^* are the minors corresponding to the matrix P, then we have, using Theorem 6:1,

$$A_h^* = (-1)^{n-h} D^{\alpha}_{\alpha_0} P^{\kappa_h}_{\alpha}, \qquad G_h = (-1)^{n+h} D^{\alpha_0}_{\beta} f^{\beta}_{\kappa_h}/m!,$$

where $\alpha_0 = (1, 2, \cdots, m)$, $\kappa_h = (1, \cdots, h-1, h+1, \cdots, n)$, and α and β are m-partite. Thus we see that

$$E_r = f(y, P) - P^{\kappa_h}_{\alpha} F^{\alpha}_{\kappa_h} = f(y, P) - P^{\iota}_{\alpha} F^{\alpha}_{\iota}/m!.$$

Consequently, for $n = m + 1$ we have $E_1 = E_c = E_r$.

The function E_1 can be exhibited in a form which more clearly displays some of its properties. This form is

$$E_1(y, p, P) = f(y, P) - f(y, p) - \sum_{g=1}^{m} \frac{1}{g!} (P - p)^{\iota}_{\alpha} F^{\alpha}_{\iota}, \tag{9:13}$$

where ι and α are now g-partite as in Section 4, $(P - p)^{\iota}_{\alpha}$ is the product of the quantities $(P^{i_\lambda}_{\alpha_\lambda} - p^{i_\lambda}_{\alpha_\lambda})$ $(\lambda = 1, \cdots, g)$, and $F^{\alpha}_{\iota} = D^{\alpha}_{\beta} f^{\beta}_{\iota}/g!$. To prove this formula, one expands the function $H(y, p, P)$ about the point (y, p, p) by Taylor's theorem and simplifies the result by employing (5:4) and (6:12).

We have already seen that $E_c = E_0$ if $P - p$ has rank one. But in this case we see that for $g > 1$ the product $(P - p)^{\iota}_{\alpha}$ is unaltered by any permutation of ι. Hence $(P - p)^{\iota}_{\alpha} F^{\alpha}_{\iota} = 0$ if $g > 1$. For $g = 1$ we have $F^{\alpha}_{i} = f^{\alpha}_{i}$, and thus we conclude from (9:13) that $E_1 = E_0 = E_c$ in case $P - p$ has rank one.

The function E_1 is not open to the objection raised against E_0 that it does not vanish identically for integrands of invariant integrals. In fact, if $f(y, p) = p^{\iota}_{\beta} C^{\beta}_{\iota}$ where ι and β are m-partite and C^{β}_{ι} is skew in ι and β, then $f^{\gamma}_{\iota} = m!\, C^{\gamma}_{\iota}$, $D^{\beta}_{\gamma} f^{\gamma}_{\iota} = (m!)^2 C^{\beta}_{\iota}$, so that we conclude from (9:11) that $E_1(y, p, P) = 0$.

This result is not true for the function E_c. To see this take $m = 2$, $n = 4$ and let f be given by

$$f = p^1_1 p^2_2 - p^2_1 p^1_2 + p^3_1 p^4_2 - p^4_1 p^3_2.$$

Then f satisfies the homogeneity condition (2:2) and is actually the integrand of an invariant integral. We find by straightforward calculation that

$$E_c = \sum_{i<k} [(P^i_1 - p^i_1)(P^k_2 - p^k_2) - (P^i_2 - p^i_2)(P^k_1 - p^k_1)] A_{ik},$$

in which $A_{ik} = p^h_1 p^j_2 - p^h_2 p^j_1$, where h and j are selected so that (i, k, h, j) is an even permutation of $(1, 2, 3, 4)$. Therefore we see that $E_c \not\equiv 0$. It follows that $E_c \geqq 0$ is not necessary for a minimum.

Similar results may be stated for the Legendre condition. The quadratic form

$$Q(y, p, v) = [f^{ab}_{ik}(y, p) - H^{ab}_{ik}(y, p, p)] v^i_a v^k_b,$$

where $H^{ab}_{ik} = \partial^2 H/\partial P^i_a \partial P^k_b$, is the Legendre quadratic form belonging to the E-function $E(y, p, P) = f(y, P) - H(y, p, P)$. As a consequence of $E_0 \geqq 0$ for all P such that $P - p$ has rank one, we conclude that

$$Q_0(y, p, v) = f^{ab}_{ik}(y, p) v^i_a v^k_b \geqq 0 \tag{9:14}$$

for all matrices v of rank one. This result has also been proved independently of the Weierstrass condition by Boerner [IV].

Using the identities of Theorem 3:2, we can prove with the help of a little computation which will be omitted that the Legendre quadratic forms for the E-functions E_1 and E_c are respectively

$$Q_1(y, p, v) = R^{ab}_{jl}(q^i_j v^i_a)(q^k_l v^k_b),$$

$$Q_c(y, p, v) = f^{-1}(fR^{ab}_{jl} + C^{ab}_{jl})(q^i_j v^i_a)(q^k_l v^k_b).$$

Therefore Q_1 and Q_c vanish identically when $v = pB$, although we see from (3:8) that this property is not possessed by Q_0. If $f = p^\iota_\beta C^\beta_\iota(y)$, where ι and β are m-partite and C^β_ι is skew in ι and β, then Q_1 vanishes identically but this need not be true for either Q_0 or Q_c.

10. The second variation

Let V be a fixed extremal variety. The second variation of $I(V)$ along V is the expression

$$J_2(\eta) = \int_T 2\omega(t, \eta, \pi)\, dT, \tag{10:1}$$

where the function 2ω was defined in equation (3:13). If V is a minimizing variety we must have $I_2(\eta) \geqq 0$ in the class of admissible variations $\eta^i(t)$ which vanish on B. The Haar-Coral accessory equations are

$$\int_{T_0} \omega_i\, dT = \int_{B_0} \omega^a_i m_a\, dB. \tag{10:2}$$

The Euler equations for $I_2(\eta)$ are called the Jacobi equations and are $J_i(\eta) = 0$, where $J_i(\eta)$ was defined in (3:14). Since V is an extremal we see from (3:16) that $p^i_a J_i(\eta) = 0$. Thus there exist functions $J^j(\eta)$ such that $J_i(\eta) = J^j(\eta) q^i_j$ so that $J_i(\eta) = 0$ if and only if $J^j(\eta) = 0$.

Assuming now that V is of class C''' we shall subject the second variation to the same transformation used in Section 8 on the first variation. We set

$$\eta^i = p^i_a Z^a + u^i_j w^j = \phi^i + u^i_j w^j, \qquad \phi^i = p^i_a Z^a.$$

Since equation (3:17) implies that $J_i(\phi) = 0$, we find after a little calculation that

$$2\omega(\eta) = 2\omega(uw) + \partial[(2u^i_j w^j + \phi^i)\omega^a_i(\phi)]/\partial t_a.$$

If $\eta^i = 0$ on B, then $w^j = \phi^i = 0$ on B, and by Green's lemma we have

$$I_2(\eta) = \int_T 2\omega(uw)\, dT.$$

If we set $u_j = (u_j^1, \cdots, u_j^n)$, some more computation shows that

$$2\omega(uw) = w^j w^l F_{jl} + w_b^j w_a^l f_{ki}^{ab} u_j^i u_l^k + \partial[w^j w^l u_l^k \omega_k^a(u_j)]/\partial t_a,$$

where $w_b^j = \partial w^j/\partial t_b$ and F_{jl} is defined by the equation

$$F_{jl} = u_l^k J_k(u_j). \tag{10:3}$$

Another application of Green's lemma suffices to prove the following

THEOREM 10:1. *The second variation* (10:1) *is expressible in the form*

$$I_2(\eta) = \int_T (w^j w^l F_{jl} + w_b^j w_a^l f_{ki}^{ab} u_j^i u_l^k)\, dT, \tag{10:4}$$

in which the functions F_{jl} are defined by (10:3) *and $w = [(q'u)^{-1}q]\eta$. The non-negativeness of* (10:4) *for arbitrary functions $w^j(t)$ of class C' which vanish on B is equivalent to the non-negativeness of* (10:1) *for arbitrary admissible variations $\eta^i(t)$ vanishing on B. The Jacobi equations are*

$$\partial(w_b^j f_{ki}^{ab} u_j^i u_l^k)/\partial t_a - \tfrac{1}{2} w^j (F_{jl} + F_{lj}) = 0. \tag{10:5}$$

Both this theorem and Theorem 8:1 are generalizations of theorems stated by Bliss [II] for the case $m = 2$, $n = 3$. When $n = m + 1$, one obtains a further simplification as is seen in the

COROLLARY. *If $n = m + 1$ and we take $u_1^i = A_i/H$, where $H^2 = A_i A_i$, then the second variation may be written in the form*

$$I_2(\eta) = \int_T (w^2 F_1 + w_a w_b R^{ab})\, dT \qquad (w = A_i \eta^i/H^2), \tag{10:6}$$

where $R^{ab} = R_{11}^{ab} H$, the quantity R_{11}^{ab} being defined by (6:13), *and where $F_1 = J^1(A/H)H$, the function $J^1(\eta)$ being determined by the equations $J_i(\eta) = J^1(\eta)A_i$. The Jacobi equation is*

$$\partial(w_a R^{ab})/\partial t_b = w F_1. \tag{10:7}$$

11. The Jacobi condition

Let us call an admissible variation $\eta^i(t)$ normal in case it satisfies in T the equations

$$p_a^i \eta^i(t) = 0, \tag{11:1}$$

where the p_a^i are the derivatives belonging to the minimizing variety V, assumed now to be of class C'. By an admissible subregion T_0 of T we mean a subregion of T whose boundary B_0 satisfies the conditions imposed on B in Section 7, and we denote by E_0 the set of intersections of the pieces of B_0. Using the methods

which Raab [XVII] employed to prove a Jacobi condition in the parametric problem for varieties of class C'' and which Carson [VI] used in the non-parametric problem for varieties of class C', we can prove the following

THEOREM 11:1. *If V is a minimizing variety of class C' on which $Q_0(y, p, v) > 0$ for all matrices v of rank one such that $p'v = 0$, then there can exist no solution $\eta^i = z^i(t)$ of class C' of the equations*

$$\int_{T'} \omega_i \, dT = \int_{B'} \omega_i^a m_a \, dB \tag{11:2}$$

for all cubes T' contained in an admissible subregion T_0, such that the variation $z^i(t)$ is normal, vanishes on B_0, and has first partial derivatives not all zero at some point t_0 of $B_0 - E_0$ in T.

For if such a solution existed, we could show precisely as in Carson [VI] that the matrix $v = (\partial z^i(t_0)/\partial t_a)$ would have rank one and would also satisfy $Q_0(y, p, v) = 0$. By hypothesis we have $p_a^i z^i = 0$. Since $z^i(t_0) = 0$, it follows that $p_a^i(t_0)v_b^i = 0$. This contradiction proves the theorem.

12. The definition of a field

By a field we mean a region $\mathfrak{F}$ of y-space on which is defined a set of slope functions $P_a^i(y)$ of class C'. Moreover, we assume the existence of an integral

$$I^*(V) = \int_T H(y, p) \, dT \tag{12:1}$$

which is invariant in the sense that it has the same value for all admissible varieties lying in $\mathfrak{F}$ and having the same boundary. The function H is also required to satisfy

$$H(y, P(y)) = f(y, P(y)), \qquad H_i^a(y, P(y)) = f_i^a(y, P(y)), \tag{12:2}$$

where $H_i^a = \partial H/\partial p_a^i$. Since every variety V is a minimizing variety for an invariant integral, and therefore satisfies the Euler-Lagrange equations for the integral, a familiar argument based on equations (12:2) suffices to show that every solution of the equations

$$\partial y^i/\partial t_a = P_a^i(y) \tag{12:3}$$

is an extremal for the integral (7:1). Such an extremal is called an extremal of the field. If V is an extremal of the field, then

$$I(V) = I^*(V). \tag{12:4}$$

13. Invariant integrals

In order to determine whether a region $\mathfrak{F}$ with given slope functions forms a field, it is necessary to determine whether or not the integral (12:1) is invariant. The following theorem is an analogue for parametric problems of the corresponding theorem for non-parametric problems proved by Landers [IX].

THEOREM 13:1. *A parametric integral* (12:1) *is invariant only if the integrand function* $H(y, p)$ *has the form*

$$H(y, p) = p^{\iota}_{\alpha} C^{\alpha}_{\iota}(y) \qquad (\iota, \alpha\ m\text{-partite}), \tag{13:1}$$

where $C^{\alpha}_{\iota}(y)$ *is skew in* ι *and* α *and satisfies the partial differential equations*

$$\partial C^{\alpha}_{\iota}/\partial y^{k} = \partial C^{\alpha}_{\iota_b}/\partial y^{i_b} \tag{13:2}$$

in which $\iota_b = (i_1, \cdots, i_{b-1}, k, i_{b+1}, \cdots, i_m)$. *These conditions are also sufficient that* $I^*(V) = I^*(V_0)$ *hold for every pair of varieties* V *and* V_0 *with the same boundary and admitting representations with the same parameters, provided that the region* $\mathfrak{F}$ *is convex.*

If we assume that $I^*(V)$ is invariant, the Euler equations

$$H_k = H^{b}_{ki} p^{i}_{b} + H^{bc}_{ki} p^{i}_{bc}$$

must be identities in y^i, p^i_a and $p^i_{ab} = p^i_{ba}$. Thus we have

$$H_k = H^{b}_{ki} p^{i}_{b} \tag{13:3}$$

and also $H^{bc}_{ki} + H^{cb}_{ki} = 0$. From this result it is seen by differentiation that if κ and β are g-partite, then $H^{\beta}_{\kappa}(y, p)$ is skew in κ and β. Therefore $H(y, p)$ must be a polynomial of degree at most m in p^i_a,

$$H(y, p) = C(y) + \sum_{g=1}^{m} p^{\kappa}_{\beta} C^{\beta}_{\kappa}(y).$$

Then the equations $H^{\beta}_{\kappa}(y, 0) = g!\, C^{\beta}_{\kappa}(y)$ show that $C^{\beta}_{\kappa}(y)$ is skew in β and κ. Since $I^*(V)$ is a parametric integral, we must have $H(y, pB) = |B| H(y, p)$ for every m-rowed square matrix B with positive determinant. Take $B = 2I$, where I is the identity matrix. Then we have as an identity in y^i, p^i_a

$$C(y) + \sum_{g=1}^{m} 2^{g} p^{\kappa}_{\beta} C^{\beta}_{\kappa}(y) = 2^{m} C(y) + \sum_{g=1}^{m} 2^{m} p^{\kappa}_{\beta} C^{\beta}_{\kappa}(y).$$

Equating coefficients shows that $C(y) = 0$, $C^{\beta}_{\kappa}(y) = 0$ for $g < m$, and thus we have proved that H must have the form specified in (13:1). For this function the equation (13:3) takes the form

$$p^{\iota}_{\alpha} \partial C^{\alpha}_{\iota}/\partial y^{k} = m p^{i\rho}_{b\sigma} \partial C^{b\sigma}_{k\rho}/\partial y^{i},$$

where $\rho = (r_1, \cdots, r_{m-1})$, $\sigma = (s_1, \cdots, s_{m-1})$, the r's and s's being (for the purposes of this section only) indices ranging from 1 to n. Equating coefficients of p^{ι}_{α} on both sides of this equation shows that (13:2) must hold.

To prove the sufficiency, we first prove two lemmas.

LEMMA 13:1. *If* $\mathfrak{F}$ *is convex and the functions* $C^{\iota}_{\alpha}(y)$ *satisfy the conditions of Theorem* 13:1, *then there exist functions* $E^{\kappa}(y)$ *with* $\kappa = (k_1, \cdots, k_{m-1})$ *such that*

$$(-1)^{c-1} \partial E^{\iota'_c}/\partial y^{i_c} = C^{\alpha_0}_{\iota} \tag{13:4}$$

holds, where $\alpha_0 = (1, \cdots, m)$, $\iota'_c = (i_1, \cdots, i_{c-1}, i_{c+1}, \cdots, i_m)$.

It can be assumed that the origin is in $\mathfrak{F}$. Let

$$V_\iota^l(y) = \sum C_\iota^{\alpha_0}(y^\rho),$$

where the sum is taken for all possible choices of the collection $\rho = (r_1, \cdots, r_l)$ of distinct integers such that ρ and ι have no component in common, and $C_\iota^{\alpha_0}(y^\rho)$ stands for the function obtained from $C_\iota^{\alpha_0}(y)$ by replacing each y^{r_λ} by 0. We next define

$$W_\iota(y) = G_l V_\iota^l(y), \qquad (l = 0, \cdots, n - m),$$

where the constants G_l are defined by

$$G_l = (n - l - 1)!/n!. \tag{13:5}$$

We now assert that the functions $E^\kappa(y)$ defined by

$$E^\kappa(y) = \int_0^{y^k} W_{k\kappa}(y)\, dy^k$$

are solutions of (13:4). To prove this we first remark that we may suppose that the components of ι in (13:4) are distinct. Since W_ι is skew in ι we then find that

$$(-1)^{c-1}\partial E^{\iota_c'}/\partial y^{i_c} = mW_\iota + \int_0^{y^k} G_l \partial V_{\iota_c}^l/\partial y^{i_c}\, dy^k,$$

where $\iota_c' = (i_1, \cdots, i_{c-1}, i_{c+1}, \cdots, i_m)$, $\iota_c = (i_1, \cdots, i_{c-1}, k, i_{c+1}, \cdots, i_m)$ and the sum on the right is for all k not components of ι. From the definition of V_ι^l and equation (13:2) we have for fixed k,

$$\begin{aligned}\int_0^{y^k} \partial V_{\iota_c}^l/\partial y^{i_c}\, dy^k &= \sum \int_0^{y^k} \partial C_{\iota_c}^{\alpha_0}(y^\rho)/\partial y^{i_c}\, dy^k \\ &= \sum \int_0^{y^k} \partial C_\iota^{\alpha_0}(y^\rho)/\partial y^k\, dy^k,\end{aligned}$$

where the second sum is taken for all ρ such that no r_λ is a component of ι or is equal to k. On summing for k not a component of ι we have

$$\begin{aligned}\int_0^{y^k} \partial V_{\iota_c}^l/\partial y^{i_c}\, dy^k &= (n - m - l)\sum C_\iota^{\alpha_0}(y^\rho) - \sum C_\iota^{\alpha_0}(y^{\rho k}) \\ &= (n - m - l)V_\iota^l - V_\iota^{l+1},\end{aligned}$$

if we agree that $V_\iota^{n-m+1} = 0$. It then follows that the left hand side of (13:4) has the value

$$nG_0 C_\iota^{\alpha_0}(y) + V^l(nG_l - lG_l - G_{l-1}) = C_\iota^{\alpha_0}(y) \qquad (l = 1, \cdots, n - m),$$

since we see from (13:5) that $G_0 = 1/n$, $nG_l - lG_l - G_{l-1} = 0$.

LEMMA 13:2. *If $H(y, p)$ has the form of Theorem* 13:1, *then there exist functions H^c such that*

$$\partial H^c/\partial t_c = H. \tag{13:6}$$

We define H^c by the equations

$$H^c = m!(-1)^{c-1}E^\kappa p^\kappa_{\beta_c}, \tag{13:7}$$

where the E^κ are the functions obtained in the previous lemma and $\beta_c = (1, \cdots, c-1, c+1, \cdots, m)$. Then we have

$$\partial H^c/\partial t_c = m!(-1)^{c-1}[\partial E^\kappa/\partial y^d p^{d\kappa}_{c\beta_c} + E^\kappa \partial p^\kappa_{\beta_c}/\partial t_c].$$

Since E^κ is skew in κ, we can write

$$m!(-1)^{c-1}E^\kappa \partial p^\kappa_{\beta_c}/\partial t_c = mE^\kappa \partial S^\kappa_c/\partial t_c,$$

where $S^\kappa_c = (-1)^{c-1}p^\kappa_\epsilon D^\epsilon_{\beta_c}$ with $\epsilon = (e_1, \cdots, e_{m-1})$ is the cofactor of $\partial/\partial t_c$ in the determinant

$$\partial S^\kappa_c/\partial t_c = \begin{vmatrix} \partial/\partial t_c \\ p_c^{k_\mu} \end{vmatrix} \qquad (\mu = 1, \cdots, m-1).$$

By a lemma of Landers [IX, p. 22], this determinant is zero. Consequently we have

$$\partial H^c/\partial t_c = m!(-1)^{c-1}\partial E^{\iota_c}/\partial y^{i_c} p^\iota_{\alpha_0} = m!C^{\alpha_0}_\iota p^\iota_{\alpha_0} = p^\iota_\alpha C^\alpha_\iota = H,$$

as desired.

To finish the proof of the theorem, let us observe that by Green's lemma, we can write

$$I^*(V) = \int_B H^c m_c \, dB, \tag{13:8}$$

where m_c are the direction cosines of the outer normal to B. In terms of the quantities S^κ_c introduced in the proof of Lemma 13:2, we have

$$H^c m_c = mE^\kappa S^\kappa_c m_c = mE^\kappa \begin{vmatrix} m_c \\ p_c^{k_\mu} \end{vmatrix}. \tag{13:9}$$

If V and V_0 are two varieties with the same boundary and admitting representation with the same parameters, then we have in a neighborhood of each point of the boundary B of T near which B is represented by functions $t_a(x_1, \cdots, x_{m-1})$ the equations $y^i(t(x)) = Y^i(t(x))$, where $y^i(t)$ and $Y^i(t)$ are the functions defining V and V_0 respectively. Differentiation of this relation shows that $(p^i_c - P^i_c)t^c_\mu = 0$, where $P^i_c = \partial Y^i/\partial t_c$ and $t^c_\mu = \partial t_c/\partial x_\mu$. There therefore exist quantities K_i such that $P^i_c = p^i_c + K_i m_c$. Then by elementary determinant theory, we conclude that the determinant in (13:9) is unaltered if we replace

$p_c^{k_\mu}$ by $P_c^{k_\mu}$. It then follows from (13:8) and (13:9) that $I^*(V)$ is equal to $I^*(V_0)$, completing the proof of the theorem.

14. The set $H(\beta; \epsilon)$

In the next section we shall define the constants $M_{\alpha;\gamma}^{\beta;\epsilon}$, originally defined in an inductive manner by (4:5), (4:6) and (4:7), explicitly in terms of a certain subset $H(\beta; \epsilon)$ of the symmetric group $G(\beta\epsilon)$ of all permutations on the components of β and ϵ. Here we shall define $H(\beta; \epsilon)$ and state without proof those properties needed for the subsequent discussion.

We recall that every permutation S in $G(\beta\epsilon)$ is expressible uniquely as the product of commutative cycles on different letters. With this fact in mind we define the set $H(\beta; \epsilon)$ to consist of all of the permutations S in $G(\beta\epsilon)$ which, when expressed as products of cycles on different letters, have no cycle containing more than one component of ϵ. For example, if $g = 1$, so that $\beta = (b)$, we find that $H(\beta; \epsilon)$ consists of the identity permutation and the h transpositions (b, e_μ) $(\mu = 1, \cdots, h)$. The structure of the set $H(\beta; \epsilon)$ is described in the following

LEMMA 14:1. *If $h = 1$, we have $H(\beta; e) = G(\beta e)$. If $g = 1$, then $H(b; \epsilon e) = H(b; \epsilon) + (b, e)$, the "+" denoting logical sum. Every permutation S in $H(\beta b; \epsilon)$ is expressible uniquely in the form RQ, where R is in $H(b; \epsilon)$ and Q is in $H(\beta; b\epsilon)$; conversely, every such product is in $H(\beta b; \epsilon)$. The set $H(\beta; \epsilon)$ contains $(g + h)!/h!$ permutations.*

15. The explicit definition of $M_{\alpha;\gamma}^{\beta;\epsilon}$

The constants $M_{\alpha;\gamma}^{\beta;\epsilon}$ may now be defined by the equation

$$M_{\alpha;\gamma}^{\beta;\epsilon} = \sum (-1)^S \delta_{\alpha\gamma}^{(\beta\epsilon)S} \tag{15:1}$$

where the sum is taken for all permutations S in $H(\beta; \epsilon)$. Then we find from equation (4:2) and the first sentence in Lemma 14:1 that

$$M_{\alpha;e}^{\beta;e} = D_{\alpha c}^{\beta e}.$$

Setting $g = 1$ in this equation we see that the constants $M_{\alpha;\gamma}^{\beta;\epsilon}$ defined explicitly by (15:1) satisfy (4:5). With the help of the second sentence in Lemma 14:1, we see that

$$M_{\alpha;\gamma c}^{\beta;\epsilon e} = \sum (-1)^S \delta_{a\gamma c}^{(b\epsilon e)S} - \delta_{a\gamma c}^{\epsilon e b},$$

where the sum is taken for all permutations S in $H(b; \epsilon)$. Consequently equation (4:6) also holds. To prove (4:7) we perform the following calculation:

$$\begin{aligned} M_{\alpha;s\sigma}^{\beta;b\epsilon} M_{a;\gamma}^{s;\sigma} &= \sum (-1)^Q (-1)^R \delta_{\alpha s\sigma}^{(\beta b\epsilon)Q} \delta_{a\gamma}^{(s\sigma)R} \\ &= \sum (-1)^{RQ} \delta_{\alpha a\gamma}^{(\beta b\epsilon)RQ}, \end{aligned}$$

where the sums are taken for all R in $H(s; \sigma)$ and $H(b; \epsilon)$, respectively, and for all Q in $H(\beta; b\epsilon)$. The third sentence in Lemma 14:1 now shows that (4:7) holds. Thus we have proved that the constants $M_{\alpha;\gamma}^{\beta;\epsilon}$ defined explicitly by (15:1) satisfy (4:5), (4:6) and (4:7), and consequently (5:4) also.

We shall now prove the identity (6:12) which we have used several times. We have from (15:1) and the obvious fact that $H^{-1}(\beta;\epsilon) = H(\beta;\epsilon)$ that

$$D^{\beta\epsilon}_{\rho\sigma} M^{\rho;\sigma}_{\alpha;\gamma} = \sum (-1)^s D^{\beta\epsilon}_{\rho\sigma} \delta^{\rho\sigma}_{(\alpha\,\gamma)s} = \sum (-1)^s D^{\beta\epsilon}_{(\alpha\,\gamma)s} = \sum D^{\beta\epsilon}_{\alpha\,\gamma} .$$

By the last sentence of Lemma 14:1, there are $(g + h)!/h!$ permutations in the set $H(\beta;\epsilon)$. Consequently (6:12) is true.

TUSKEGEE INSTITUTE.

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ON THE MAXIMUM OF THE FUNDAMENTAL FUNCTIONS OF THE ULTRASPHERICAL POLYNOMIALS

BY P. ERDÖS

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In the present note we are going to prove the following theorem: Let $-1 \leqq x_1 < x_2 \cdots < x_n \leqq 1$ be the roots of the ultraspherical polynomial $P_n^{(\alpha)}(x)$ with $0 \leqq \alpha \leqq 3/2$. (The normalisation is of no importance.) $\alpha = \frac{1}{2}$ gives the Legendre polynomial $\alpha = 3/2$ gives $U_n(x) = T'_{n+1}(x)$, where $T_n(x)$ is the n^{th} Tchebicheff polynomial. Let

$$l_k^{(n)}(x) = \frac{P_n^{(\alpha)}(x)}{P_n'^{(\alpha)}(x_k)(x - x_k)}$$

be the fundamental polynomial of the Lagrange interpolation. Then

$$\max_{k=1,2,\cdots n,\ -1 \leqq x \leqq 1} | l_k^{(n)}(x) | = l_1^{(n)}(-1) = l_n^{(n)}(1).$$

Special cases of this theorem have been proved by Erdös-Grünwald[1] and Webster[2] (the cases $\alpha = 1/2$ and $\alpha = 3/2$). If there is no danger of confusion we shall omit the upper index n in $l_k^{(n)}(x)$.

PROOF OF THE THEOREM. It clearly suffices to consider the $l_k(x)$ with $-1 \leqq x_k \leqq 0$. From the differential equation of the ultraspherical polynomials[3] we obtain

$$l_k'(x_k) = \frac{P_n'^{(\alpha)}(x_k)}{\frac{1}{2}P_n'^{(\alpha)}(x_k)} = \frac{\alpha x_k}{1 - x_k^2}. \tag{1}$$

Thus for $x_k \leqq x \leqq x_{k+1}$ $0 \leqq l_k(x) \leqq 1$. Suppose now that $k \neq 1$, then we prove that in (x_{k-1}, x_k) $l_k(x)$ lies below its tangent at x_k. Denote by $y_1, y_2, \cdots y_{n-1}$ the roots of $l_k'(x)$ and by $z_1, z_2, \cdots z_{n-2}$ the roots of $l_k''(x)$. From (1) it follows that $x_{k-1} < y_{k-1} < x_k$. To prove our assertion it suffices to show that $z_{k-1} > x_k$. First we prove that $y_{k-1} > \dfrac{x_{k-1} + x_k}{2} = u$. From (1)

$$\frac{1}{2}\frac{\alpha}{1 + x_k} + \sum_{i<k} \frac{1}{x_k - x_i} = \frac{\alpha}{2(1 - x_k)} + \sum_{j>k} \frac{1}{x_j - x_k},$$

thus

$$\frac{1}{1 + x_k} + \sum_{i<k} \frac{1}{x_k - x_i} > \sum_{j>k} \frac{1}{x_j - x_k}. \tag{2}$$

[1] ERDÖS-GRUNWALD, Bull. Amer. Math. Soc. 44 (1938), p. 515-518.

[2] WEBSTER, ibid. 45 (1939), p. 870-873.

[3] See e.g. G. SZEGÖ, *Orthogonal Polynomials*, Amer. Math. Soc. Coll. Publications vol. XXIII p. 59. Our notation differs from that of Szegö. This α has to be replaced by $\alpha + 1$.

Now from (2)

$$\sum_{i<k}\frac{1}{u-x_i}=\sum_{i<k-1}\frac{1}{u-x_i}+\frac{1}{u-x_{k-1}}>\sum_{i<k-1}\frac{1}{x_k-x_i}+\frac{1}{u-x_{k-1}}$$

$$=\sum_{i<k}\frac{1}{x_k-x_i}-\frac{1}{x_k-x_{k-1}}+\frac{1}{u-x_{k-1}}$$

$$=\sum_{i<k}\frac{1}{x_k-x_i}+\frac{1}{x_k-x_{k-1}}>\sum_{j>k}\frac{1}{x_j-x_k}-\frac{1}{x_k+1}$$

$$+\frac{1}{x_k-x_{k-1}}>\sum_{j>k}\frac{1}{x_j-x_k}>\sum_{j>k}\frac{1}{x_j-u}$$

which proves $y_{k-1}>u$. Now evidently from $y_{k-1}>u$

$$\sum_{i<k}\frac{1}{x_k-y_i}=\sum_{i<k-1}\frac{1}{x_k-y_i}+\frac{1}{x_k-y_{k-1}}>\sum_{i<k-1}\frac{1}{x_k-x_i}+\frac{1}{x_k-u}$$

$$=\sum_{i<k}\frac{1}{x_k-x_i}-\frac{1}{x_k-x_{k-1}}+\frac{1}{x_k-u}$$

$$=\sum_{i<k}\frac{1}{x_k-x_i}+\frac{1}{x_k-x_{k-1}}>\sum_{i<k}\frac{1}{x_k-x_i}+\frac{1}{x_k+1}$$

and

$$\sum_{j\geqq k}\frac{1}{y_j-x_k}<\sum_{j>k}\frac{1}{x_j-x_k}.$$

Thus by (2)

$$\sum_{i\leqq k-1}\frac{1}{x_k-y_i}>\sum_{j\geqq k}\frac{1}{y_j-x_k},$$

which proves $z_{k-1}>x_k$.

Thus we obtain for $k\neq 1$

$$\max_{x_{k-1}\leqq x\leqq x_{k+1}}|l_k(x)|<1+\frac{\alpha|x_k|}{1+|x_k|} \tag{3}$$

and of course from (1)

$$l_1(-1)>1+\frac{\alpha|x_k|}{1+|x_1|}\geqq 1+\frac{\alpha|x_k|}{1+|x_k|}. \tag{4}$$

Suppose now $1/2\leqq\alpha\leqq 3/2$. A well known theorem of M. Riesz[4] states: Let $f(x)$ be a polynomial of degree n which assumes its absolute maximum in $(-1,1)$ at x_0; then for every root x_k of $f(x)$ in $(-1,+1)$ we have $\vartheta_k-\vartheta_0\geqq\frac{\pi}{2n}$. Here $x_k=\cos\vartheta_k$, $x_0=\cos\vartheta_0$, $0<\vartheta_k\leqq\pi$, $0<\vartheta_0\leqq\pi$.

[4] M. Riesz, Jahresbericht der Deutschen Math Vereinigung, (1916) p. 354-368.

Let $-1 \leqq x_1 < x_2 < \cdots < x_n \leqq 1$ be the roots of $P_n^{(\alpha)}(x)$; put $\cos\vartheta_k = x_k$ $0 < \vartheta_k < \pi$, then it is well known that[5]

$$\vartheta_v - \vartheta_{v+1} \leqq \frac{\pi}{n + (2\alpha + 1)/2} \leqq \frac{\pi}{n}.$$

Thus $| l_k(x) |$ can take its absolute maximum in $(-1, 1)$ only in (x_{k-1}, x_{k+1}), or at the points -1 and 1. We shall prove that for $k \neq 1$,

$$| l_k(-1) | < l_1(-1). \tag{5}$$

It clearly suffices to show that

$$| P_n^{(\alpha)\prime}(x_k)(1 + x_k) | > | P_n^{(\alpha)\prime}(x_1)(1 + x_1) |.$$

Or that

$$| P_n^{(\alpha)\prime}(x_k)(1 - x_k^2) | \geqq | P_n^{(\alpha)\prime}(x_1)(1 - x_1^2) |. \tag{6}$$

By the differential equation we have

$$(1 - x^2)P_n^{(\alpha)\prime\prime}(x) - (2\alpha + 4)xP_n^{(\alpha)\prime}(x) + n(n + 2\alpha + 3)P_n^{(\alpha)}(x) = 0.$$

Now apart from a constant factor $P_n^{(\alpha)\prime}(x) = P_{n-1}^{(\alpha+1)}(x)$. Thus we can write

$$(1 - x^2)P_n^{(\alpha)\prime}(x) + c_1xP_n^{(\alpha)}(x) + c_2P_{n-1}^{(\alpha-1)}(x) = 0.$$

Hence for the roots of $P_n^{(\alpha)}(x)$

$$| (1 - x_k^2)P_n^{(\alpha)\prime}(x_k) | = | c_2P_{n+1}^{(\alpha-1)}(x_k) |.$$

The points x_k are the relative maxima of $P_{n+1}^{\alpha-1}(x)$. It is well known[6] that for $\alpha \leqq 1/2$ the successive maxima of $P_n^{(\alpha)}(x)$ increase toward the origin i.e. for $\alpha \leqq 3/2$

$$| P_{n+1}^{(\alpha-1)}(x_1) \leqq | P_{n+1}^{(\alpha-1)}(x_k) |.$$

This proves (6) and therefore (5). By the symmetry of the x it follows that for $k \neq n$

$$l_1(-1) = l_n(1) > | l_k(1) |. \tag{7}$$

Thus, finally, from (3), (4), (6) and (7) we obtain our theorem for $1/2 \leqq \alpha \leqq 3/2$.

Suppose now that $0 \leqq \alpha < 1/2$. Then it is well known that $\vartheta_1 \leqq 2n$.[7] Thus according to the theorem of M. Riesz it suffices to consider the interval (x_1, x_n). Suppose then that $l_k(x)$ assumes its absolute maximum at x_0, and that x_0 is not in (x_{k-1}, x_{k+1}). It is easy to see that[8]

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[5] G. Szegö, ibid. p. 121, theorem 6.3.1.

[6] Ibid. p. 163 164, proof of theorem 7.32.1.

[7] Ibid. p. 117, theorem 6.21.1. $\vartheta_1 \leqq \frac{\pi}{2n}$ follows from the remark that in case of $T_n(x)(\alpha = -\frac{1}{2})\vartheta_1 = \frac{\pi}{2n}$.

[8] Erdös-Turan, Annals of Math. vol. 41 (1940) p. 429 lemma IV.

$$l_i(x_0) + l_{i+1}(x_0) > 1, \qquad x_i < x_0 < x_{i+1}.$$

According to a formula of Fejér[9]

$$\sum_{k=1}^{n} v_k(x_0) l_k^2(x_0) = 1, \quad \text{where} \quad v_k(x_k) = 1,$$

$$v_k\left(x_k + \frac{1 - x_k^2}{2\alpha x_k}\right) = 0, \qquad v_k(x) \text{ linear}, \tag{8}$$

hence

$$v_i(x_0) l_i^2(x_0) + v_{i+1}(x_0) l_{i+1}^2(x_0) + v_k(x_0) l_k^2(x_0) \leqq 1.$$

Thus from (8)

$$v_j(x_0) > 1 - \frac{2\alpha x_j}{1 + |x_j|} \geqq 1 - \frac{2\alpha |x_1|}{1 + |x_1|} = c, \qquad \tfrac{1}{2} < c \leqq 1$$

Clearly one of the numbers $v_i(x_0)$, $x_{i+1}(x_0)$ is greater than 1. Thus

$$v_i(x_0) l_i^2(x_0) + v_{i+1} l_{i+1}^2(x_0) > \min_{x+y=1, x, y>0} (x^2 + cy^2) = \frac{c}{1 + c}.$$

Hence

$$|l_k(x_0)| < \sqrt{\frac{1}{c(1 + c)}}.$$

From (4) we have

$$l_1(-1) > \frac{3 - c}{2},$$

and it is easy to see that

$$\frac{3 - c}{2} > \sqrt{\frac{1}{c(1 + c)}} \qquad (1/2 < c \leqq 1)$$

which completes the proof.

If $\alpha > 3/2$ our theorem does not hold any more, since it is easy to see that $l_1(-1)$ remains bounded but $\max l_k(x)$ does not remain bounded.

Webster[10] proved that

$$l_1^{(n)}(-1) \to (1/2j_1)^{\alpha - 2} \, |\, \Gamma(\alpha) y_\alpha(j_1) \,|^{-1},$$

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[9] L. FEJÉR, Math. Annalen, 106, (1932) p. 4 and p. 43.

[10] WEBSTER, Bull. Amer. Math. Soc. 47 (1941), p. 73.

where j_1 is the first zero of $J_{\alpha-1}$ ($J(x)$ denotes Bessel functions). I think it can be shown that

$$l_1^{(n)}(-1) < (1/2j_1)^{\alpha-2}\Gamma(\alpha)y_\alpha(j_1)\,|^{-1},$$

in fact $l_1^{(n)}(-1) < l_1^{(n+1)}(-1)$. If so, we could state the following theorem:

Let $0 \leqq \alpha \leqq 3/2$. Then

$$\max_{k=1,2,\cdots n,-1\leq x\leq 1} |\,l_k(x)\,| < (\tfrac{1}{2}j_1)^{\alpha-2}\,|\,\Gamma(\alpha)y_\alpha(j_1)\,|^{-1},$$

and this result is the best possible.

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$$l_i(x_0) + l_{i+1}(x_0) > 1, \qquad x_i < x_0 < x_{i+1}.$$

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$$\sum_{k=1}^{n} v_k(x_0) l_k^2(x_0) = 1, \quad \text{where} \quad v_k(x_k) = 1,$$
$$v_k\left(x_k + \frac{1 - x_k^2}{2\alpha x_k}\right) = 0, \qquad v_k(x) \text{ linear}, \tag{8}$$

hence

$$v_i(x_0) l_i^2(x_0) + v_{i+1}(x_0) l_{i+1}^2(x_0) + v_k(x_0) l_k^2(x_0) \leqq 1.$$

Thus from (8)

$$v_j(x_0) > 1 - \frac{2\alpha x_j}{1 + |x_j|} \geqq 1 - \frac{2\alpha |x_1|}{1 + |x_1|} = c, \qquad \tfrac{1}{2} < c \leqq 1$$

Clearly one of the numbers $v_i(x_0)$, $x_{i+1}(x_0)$ is greater than 1. Thus

$$v_i(x_0) l_i^2(x_0) + v_{i+1} l_{i+1}^2(x_0) > \min_{x+y=1, x, y>0} (x^2 + cy^2) = \frac{c}{1+c}.$$

Hence

$$|l_k(x_0)| < \sqrt{\frac{1}{c(1+c)}}.$$

From (4) we have

$$l_1(-1) > \frac{3-c}{2},$$

and it is easy to see that

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which completes the proof.

If $\alpha > 3/2$ our theorem does not hold any more, since it is easy to see that $l_1(-1)$ remains bounded but max $l_k(x)$ does not remain bounded.

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[9] L. FEJÉR, Math. Annalen, 106, (1932) p. 4 and p. 43.

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where j_1 is the first zero of $J_{\alpha-1}$ ($J(x)$ denotes Bessel functions). I think it can be shown that

$$l_1^{(n)}(-1) < (1/2j_1)^{\alpha-2}\Gamma(\alpha)y_\alpha(j_1)\,|^{-1},$$

in fact $l_1^{(n)}(-1) < l_1^{(n+1)}(-1)$. If so, we could state the following theorem: Let $0 \leqq \alpha \leqq 3/2$. Then

$$\max_{k=1,2,\cdots n,\,-1\leqq x\leqq 1} |\,l_k(x)\,| < (\tfrac{1}{2}j_1)^{\alpha-2}\,|\,\Gamma(\alpha)y_\alpha(j_1)\,|^{-1},$$

and this result is the best possible.

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ON A CLASS OF FUNCTIONS SUGGESTED BY THE ZETA OF RIEMANN

BY S. HURWITZ

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1. Introduction

Considering the prominence of $\zeta(s) = \sum n^{-s}$ in analysis, and the association of $\sum n^{-p}$ with $\sum n^{-1}(\log n)^{-p}$, $\sum (n \log n)^{-1}(\log \log n)^{-p}$, $\cdots$, in the "logarithmic scale," it is rather surprising to find nothing in the literature about the expressions $\sum n^{-1}(\log n)^{-s}$, $\sum (n \log n)^{-1}(\log \log n)^{-s}$, $\cdots$, as functions of the complex variable $s = \sigma + it$. This paper is concerned with some theorems about this class of functions.

After defining $\zeta_p(s)$ in §2, we consider its domain of regularity. That it is regular everywhere but at $s = 1$ can be established in various ways. The method of §3 seems to be the simplest, and at the same time provides us with the very useful formula, (7), for $\zeta_p(s)$.

The integrals introduced in sections 4 and 5 lead us to the Laurent expansion for $\zeta^*_{p-1}(1 - z)$. From this we deduce that $\zeta_p(s)$ has an infinite number of negative zeros. The argument, however, merely shows the existence of these zeros but gives no indication of their numerical values.

The last section is concerned with theorems on the asymptotic behavior of $\zeta_p(s)$. It is shown that on a circle with center at $s = 0$ and sufficiently large radius, r, the maximum of $|\zeta_p(s)|$ lies between $e^{r^{1-\epsilon}}$ and $e^{r^{1+\epsilon}}$; on a vertical line to the right of $\sigma = 1$, $\zeta_p(s)$ is bounded; on $\sigma = 1$, $\zeta_p(s) = O(\log_{p+1} t)$ but $\zeta_p(s) > A \log_{p+2} t$ for an arbitrarily large t; on a vertical line to the left of $\sigma = 1$, $\zeta_p(s) = O(\log_p t)^{1-\sigma}$.

2. Definitions

Let p be a non-negative integer. We understand

$$\log_0 x = x, \quad \text{and} \quad \log_{p+1} x = \log (\log_p x); \tag{1}$$

$$e_0(x) = x, \quad \text{and} \quad e_{p+1}(x) = \exp (e_p(x)); \tag{2}$$

$$f_0(x) = 1, \quad \text{and} \quad f_{p+1}(x) = [\textstyle\prod_0^p (\log_i x)]^{-1}; \tag{3}$$

$$g_p(x, s) = f_p(x)(\log_p x)^{-s}. \tag{4}$$

For convenience we shall use U, V, and W in place of $[e_{p-1}(0)]$, $[e_p(0)]$, and $[e_{p+1}(0)]$ respectively, where $[x]$, as usual, means the integral part of x.

We define $\zeta_p(s)$ as follows:

$$\zeta_p(s) = \sum_{n=V+1}^{\infty} g_p(n, s). \tag{5}$$

We note that $\zeta_0(s) = \zeta(s)$. By $(\log_p x)^{-s}$ we mean $e^{-s \log_{p+1} x}$ and always take that branch of the logarithm which is real along the positive half of the x-axis. Since

$\log_p (V+1) > \log_p (e_p(0)) = 0$, all of the logarithms involved in (5) are real numbers.

$\zeta_p(s)$ is defined by (5) only for $\sigma > 1$. The series converges absolutely if $\sigma > 1$ and uniformly in any region $\sigma > 1 + \delta$ with $\delta > 0$. By Weierstrass' theorem on a uniformly converging series of analytic functions $\zeta_p(s)$ is analytic in any such region.

In the sections that follow $p > 0$.

3. The domain of regularity

THEOREM I. *$\zeta_p(s)$ is a single-valued meromorphic function whose only singularity in the finite part of the s-plane is a simple pole with residue 1 at $s = 1$. Furthermore, in any bounded domain not containing $s = 1$ in its interior, we have uniformly*

$$\zeta_p(s) = \lim_{N\to\infty} \left\{ \sum_{n=V+1} g_p(n, s) + \frac{(\log_p N)^{1-s}}{s-1} \right\}, \tag{6}$$

or

$$\zeta_p(s) = \sum_{n=V+1}^{N} g_p(n, s) + \frac{(\log_p N)^{1-s}}{s-1} + R_N, \tag{7}$$

where

$$R_N = \int_N^\infty (x) g_p'(x, s)\, dx \tag{8}$$

and $(x) = x - [x]$.

If $\sigma > 1$

$$\zeta_p(s) = \lim_{X\to\infty} \sum_{x=V+1}^{[X]} g_p(x, s). \tag{9}$$

Now

$$\begin{aligned}
\sum_{V+1}^{[X]} g_p(x, s) &= \sum_{V+1}^{N} g_p(x, s) - N g_p(N, s) + N[g_p(N, s) - g_p(N+1, s)] \\
&\quad + (N+1)[g_p(N+1, s) - g_p(N+2, s)] + \cdots \\
&\quad + [X][g_p([X], s) - g_p(X, s)] + [X] g_p(X, s) \\
&= \sum_{V+1}^{N} g_p(x, s) - N g_p(N, s) + \int_N^{N+1} - [x] g_p'(x, s)\, dx \\
&\quad + \int_{N+1}^{N+2} - [x] g_p'(x, s)\, dx + \cdots \\
&\quad + \int_{[X]}^{X} - [x] g_p'(x, s)\, dx + [X] g_p(X, s) \\
&= \sum_{V+1}^{N} g_p(x, s) - N g_p(N, s) + \int_N^X - [x] g_p'(x, s)\, dx + [X] g_p(X, s).
\end{aligned}$$

Hence by (9), if $\sigma > 1$

$$\zeta_p(s) = \sum_{V+1}^{N} g_p(x, s) - Ng_p(N, s) + \int_N^\infty - [x]g_p'(x, s)\,dx$$

$$= \sum_{V+1}^{N} g_p(x, s) - Ng_p(N, s) + \int_N^\infty (x)g_p'(x, s)\,dx - \int_N^\infty xg_p'(x, s)\,dx$$

$$= \sum_{V+1}^{N} g_p(x, s) - Ng_p(N, s) + \int_N^\infty (x)g_p'(x, s)\,dx$$

$$- \left\{[xg_p(x, s)]_N^\infty - \int_N^\infty g_p(x, s)\,dx\right\}$$

$$= \sum_{V+1}^{N} g_p(x, s) - Ng_p(N, s) + \int_N^\infty (x)g_p'(x, s)\,dx + Ng_p(N, s) + \frac{(\log_p N)^{1-}}{s-1}$$

$$= \sum_{V+1}^{N} g_p(x, s) + \frac{(\log_p N)^{1-s}}{s-1} + \int_N^\infty (x)g_p'(x, s)\,dx.$$

Now for any s we can find an x_0 such that $| g_p'(x, s) | < x^{-1-\delta}$ with $\delta > 0$ for all $x > x_0$. $\int_N^\infty (x)g_p'(x, s)\,dx$ therefore converges for all values of s, and uniformly in any circle $| s | < R$. By choosing N sufficiently large we can make the modulus of the integral as small as we please. Since

$$(s-1)\zeta_p(s) = \lim_{N\to\infty}\left\{\sum_{n=V+1}^{N} (s-1)g_p(n, s) + (\log_p N)^{1-s}\right\}$$

can be written as an infinite series of analytic functions which converges uniformly in any circle $| s | < R$, it follows that $(s-1)\zeta_p(s)$ is an integral function. An examination of (7) shows that at $s = 1$, $\zeta_p(s)$ has a simple pole with residue 1.

4. A definite integral for $\zeta_p(s)$

We have

$$\zeta_{p-1}(1+s) = \sum_{n=U+1}^{\infty} g_{p-1}(n, 1+s) = \sum_{n=U+1}^{\infty} f_p(n)(\log_{p-1} n)^{-s}. \tag{10}$$

Hence

$$\zeta_{p-1}(1+s) - \sum_{n=U+1}^{V} g_{p-1}(n, 1+s) = \sum_{n=V+1}^{\infty} f_p(n)(\log_{p-1} n)^{-s}. \tag{11}$$

Let $\zeta_{p-1}^*(1+s)$ designate the left member of (11).

If we compare corresponding terms of the Dirichlet series for $\zeta_p(s)$ and $\zeta_{p-1}^*(1+s)$, we see that they have identical coefficients while the coefficient of $-s$ in the exponent of e in the first is the logarithm of the corresponding number

in the second. The connection between two such functions is well known.[1] We have, for $\sigma > 1$

$$\Gamma(s)\zeta_p(s) = \int_0^\infty y^{s-1}\zeta_{p-1}^*(1+y)\,dy. \tag{12}$$

5. A contour integral for $\zeta_p(s)$

As with $\zeta(s)$ we can use (12) to get a contour integral expression for $\zeta_p(s)$. We have

$$\frac{\zeta_p(s)}{\Gamma(1-s)} = \frac{1}{2\pi i}\int_C z^{s-1}\zeta_{p-1}^*(1-z)\,dz, \tag{13}$$

where the contour C starts at $-\infty$ on the negative real axis, encircles the origin once in a positive direction, and returns to its starting point. The formula is established from (12) in the case $\sigma > 1$ by shrinking the contour C into the real axis described twice and taking into consideration the different values of the logarithm on the two parts.[2]

But the right member of (13) converges for all s and uniformly in any circle $|s| < \Delta$. This will be demonstrated if

$$\left|\int_R^{-\infty} z^{s-1}\zeta_{p-1}^*(1-z)\right|$$

can be made arbitrarily small, by a proper choice of R, for all s with $|s| < \Delta$. (R is positive and the path of integration is along the x-axis.)

Now, if $z = re^{\pm\pi i}$, $|z^s| = e^{\sigma\log r \pm t\pi} \leqq r^\sigma e^{|t|\pi}$. Also[3]

$$\begin{aligned}\zeta_{p-1}^*(1+r) &< g_{p-1}(V+1, 1+r) + \int_{V+1}^\infty g_{p-1}(x, 1+r)\,dx \\ &= g_{p-1}(V+1, 1+r) + r^{-1}e^{-r\log_p(V+1)} \\ &< e^{-r\log_p(V+1)}, \quad \text{if} \quad r > 2.\end{aligned}$$

Therefore the modulus of the integrand is less than $r^{\Delta-1}e^{\Delta\pi - r\log_p(V+1)}$, and for big r, will therefore be less than e^{-Ar} with $A > 0$. Consequently, by choosing R sufficiently large, the modulus of the integral in question can be made arbitrarily small.

Since the contour integral is uniformly convergent in any finite region, it represents an integral function of s. But, if s is an integer, the integrand in (13)

[1] See, for instance, O. PERRON, Crelle's Journal, vol. 134, 1908.

[2] The method is Riemann's. For details see Ingham's Cambridge Tract on the Prime Number Theorem, pp. 41–43.

[3] For positive s, $g_p(x, s)$ is certainly a decreasing function when $x \geqq V + 1$.

is a single-valued function of z and $\zeta_p(s)/\Gamma(1-s)$ is its residue at $z=0$. We thus obtain the Laurent expansion

$$\zeta^*_{p-1}(1-z) = -\frac{1}{z} + \sum_{n=0}^{\infty} \frac{\zeta_p(-n)}{n!} z^n. \tag{14}$$

From (6) it is clear that $\zeta_p(-n)$ is a real number.

6. Real zeros of $\zeta_p(s)$

THEOREM 2. *$\zeta_p(s)$ has an infinite number of negative zeros.*

The argument is by induction. The theorem is true for $\zeta_0(s) = \zeta(s)$. We shall show that its truth for $\zeta_{p-1}(s)$ implies its truth for $\zeta_p(s)$.

Now $\zeta_p(s)$ will have an infinite number of negative zeros if the right member of (14) has an infinite number of variations of sign. Suppose then that there are only a finite number of variations in the right member of (14). Then, given any positive number G, we could find a positive number x_0 such that, for all positive z's greater than x_0 the modulus of $\zeta^*_{p-1}(1-z)$ is greater than G. But by the induction hypothesis there exist such positive z's for which $\zeta_{p-1}(1-z) = 0$, and since $0 < \log_{p-1} x \leqq 1$ if $e_{p-1}(0) < x \leqq e_p(0)$, for such z's

$$\left| \sum_{U+1}^{V} g_{p-1}(n, 1-z) \right| \leqq V - U,$$

and hence $|\zeta^*_{p-1}(1-z)| \leqq V - U$. The contradiction proves the theorem.

7. The asymptotic behavior of $\zeta_p(s)$

Let $M(r)$ represent the maximum of $|\zeta_p(s)|$ on the circle $|s| = r$.

THEOREM 3. *For sufficiently large r, we have*

$$e^{r^{1-\epsilon}} < M(r) < e^{r^{1+\epsilon}}. \tag{15}$$

The first half of (15) follows from

$$M(r) \geqq \zeta_p(r) > g_p(V+1, r) > e^{r^{1-\epsilon}}.$$

To establish the second half of (15), let $r > e_p(0)$ and take $N = [e^r] + 1$ in (7). We have

$$\left| \int_N^{\infty} (x) g'_p(x, s)\, dx \right| = \left| \int_N^{\infty} (x) g_p(x, s) \left\{ -\sum_{i=1}^{p} f_i(x) - s f_{p+1}(x) \right\} dx \right|$$

$$< (p+1) \left| \int_N^{\infty} \frac{(x)}{x} g_p(x, s)\, dx \right|$$

$$< (p+1) \int_1^{\infty} \frac{(\log x)^r}{x^2}\, dx = (p+1)\Gamma(1+r) < e^{r^{1+\epsilon_1}}.$$

Also

$$\left| \sum_{n=V+1}^{N} g_p(n, s) \right| < e^r \cdot e^{r^{1+\epsilon_2}} < e^{r^{1+\epsilon_3}},$$

and

$$\left|\frac{(\log_p N)^{1-s}}{s-1}\right| < e^{r^{1+\epsilon}}.$$

Hence for sufficiently large r, $M(r) < e^{r^{1+\epsilon}}$.

THEOREM 4. *If $\sigma > 1$, $|\zeta_p(s)| \leqq \zeta_p(\sigma)$ for all t, and $|\zeta_p(s)| > (1-\epsilon)\zeta_p(\sigma)$ for indefinitely large values of t.*

The first part follows from

$$|\zeta_p(s)| = \left|\sum_{n=V+1}^{\infty} g_p(n, s)\right| \leqq \sum_{n=V+1}^{\infty} g_p(n, \sigma) = \zeta_p(\sigma).$$

The second part follows from the Bohr-Landau argument[4] which yields the analogous result for $\zeta(s)$. By using a theorem of Dirichlet it can be shown that for given positive integers N and q and a positive number τ, there is a number t in the range $\tau \leqq t \leqq \tau q^{N-V}$ for which

$$|\zeta_p(s)| > \left\{\cos\frac{2\pi}{q} - 2(\log_p N)^{1-\sigma}\right\}\zeta_p(\sigma). \tag{16}$$

The result follows if q and N are sufficiently large.

COROLLARY. *$\zeta_p(s)$ is unbounded in any region $1 < \sigma < 1+\delta$, $t > t_0$.*

THEOREM 5. *However large t_0 may be, there are values of s in the region $1 < \sigma < 1+\delta$, $t > t_0$, for which $|\zeta_p(s)| > A \log_{p+2} t$.*

From (16) is follows that there is a value of t between 1 and 6^{N-V} for which

$$|\zeta_p(s)| > \frac{\frac{1}{2} - 2(\log_p N)^{1-\sigma}}{\sigma - 1}.$$

Choose N so that $N - 1 < e_p(8^{1/\sigma-1}) \leqq N$. Hence

$$|\zeta_p(s)| \geqq \frac{1}{4(\sigma-1)} > \frac{\log_{p+1}(N-1)}{4\log 8} > A_1 \log_{p+1} N$$

for a value of t such that $N > A_2 \log t$. Therefore $|\zeta_p(s)| > A \log_{p+2} t$.

If $\sigma - 1$ is sufficiently small, the value of t in question must be greater than any assigned to t_0; otherwise it would follow that $\zeta_p(s)$ was unbounded in the region $1 < \sigma < 1+\delta$, $1 < t < t_0$.

THEOREM 6. *$\zeta_p(s) = O(\log_{p+1} t)$ uniformly in any region $1 - A/\log_{p+1} t \leqq \sigma \leqq 2$. In particular $\zeta_p(1+it) = O(\log_{p+1} t)$.*

In the region considered, if $W < n \leqq t$,

$$|g_p(n, s)| \leqq f_p(n) \exp\{-(1 - A/\log_{p+1} t)\log_{p+1} n\} \leqq e^A g_p(n, 1),$$

while, if $V+1 \leqq n \leqq W$, $|g_p(n, s)| \leqq g_p(n, 2)$. In (7), take $N = [t]$. We have

$$\left|\sum_{n=V+1}^{N} g_p(n, s)\right| \leqq \sum_{n=V+1}^{W} g_p(n, 2) + e^A \sum_{n=W+1}^{N} g_p(n, 1)$$

$$\leqq O(1) + e^A \int_W^N g_p(x, 1)\, dx$$

$$= O(\log_{p+1} t),$$

[4] Göttinger Nachrichten (1910), 303–330.

also

$$\left|\frac{(\log_p N)^{1-s}}{s-1}\right| \leqq \frac{e^A}{|s-1|} = O(1),$$

and

$$\left|\int_N^\infty (x) g_p'(x, s)\, dx\right| \leqq O(1) + t\int_N^\infty \frac{dx}{x^2} = O(1).$$

Hence $\zeta_p(s) = O(\log_{p+1} t)$.

THEOREM 7. *$\zeta_p(1 + it)$ is unbounded.*

According to a theorem of Lindelof, if a function $f(s)$ is regular and of the form $O(t^A)$ in the half-strip $\sigma_1 \leqq \sigma \leqq \sigma_2$, $t \geqq t_0 > 0$ and $|f(s)| \leqq M$ on the whole boundary of the half-strip, then $|f(s)| \leqq M$ throughout the half-strip.

Since $\zeta_p(2 + it)$ is bounded, and by the preceding theorem $\zeta_p(s) = O(t^A)$ in the half-strip $1 \leqq \sigma \leqq 2$, $t \geqq t_0 > 0$, $\zeta_p(s)$ would be bounded in the half-strip if $\zeta_p(1 + it)$ were bounded.

THEOREM 8. *However large t_0 may be, there are values of t greater than t_0 for which*

$$|\zeta_p(1 + it)| > A \log_{p+2} t.$$

It follows from Lindelof's theorem that if $f(s)$ is regular and of the form $O(t^A)$ in the half-strip $\sigma_1 \leqq \sigma \leqq \sigma_2$, $t \geqq t_0$, and if $f(s) \to 0$ on both the lines $\sigma = \sigma_1$, $\sigma = \sigma_2$ as $t \to \infty$ then $f(s) \to 0$ uniformly in the half-strip.

By theorem 5, $f(s) = \zeta_p(s)/\log_{p+2} s$ does not tend to zero uniformly for $1 < \sigma < 2$ as $t \to \infty$ and therefore $f(1 + it)$ does not.

THEOREM 9. *If $\sigma_0 < 1$, $\zeta_p(\sigma_0 + it) = O(\log_p t)^{1-\sigma_0}$.*

In (7) take $N = [t]$ and let $[e_p(|\sigma_0|)] = \alpha$. Then

$$\left|\sum_{n=V+1}^{N} g_p(n, s)\right| \leqq \sum_{n=V+1}^{\alpha+1} g_p(n, \sigma_0) + \sum_{n=\alpha+2}^{N} g_p(n, \sigma_0)$$

$$< O(1) + \int_{\alpha+1}^{N} g_p(x, \sigma_0)\, dx$$

$$< O(1) + \frac{(\log_p N)^{1-\sigma_0}}{1-\sigma_0} = O(\log_p t)^{1-\sigma_0},$$

$$\left|\frac{(\log_p N)^{1-s}}{s-1}\right| \leqq \frac{(\log_p t)^{1-\sigma_0}}{t} = O(1),$$

and $|R_n| = O(1)$. Hence $\zeta_p(\sigma_0 + it) = O(\log_p t)^{1-\sigma_0}$.

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ON A CLASS OF NÖRLUND MATRICES

BY L. L. SILVERMAN AND O. SZASZ

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1. Introduction

If in the Nörlund transformation

$$P_n y_n = \sum_{\nu=0}^{n} p_\nu x_{n-\nu}, \qquad P_n = \sum_{0}^{n} p_\nu,$$

the first k p's are set equal to *unity* and the remaining p's are *zero*, the transformation becomes

$$ky_n = \sum_{\nu=0}^{k-1} x_{n-\nu}, \qquad n \geqq k-1,$$

and will be denoted by Z_k. Special cases for $k = 2$ and 3 are respectively[1]

$$y_n = \frac{x_n + x_{n-1}}{2}, \quad \textit{and} \quad y_n = \frac{x_n + x_{n-1} + x_{n-2}}{3}.$$ [2]

The transformation Z_k leads us to the examination of matrices of finite rank, that is, triangular matrices all of whose elements below a given diagonal are equal to zero. The relationship of these matrices to one another and to triangular matrices in general is studied. From the standpoint of summability a study is also made of the properties of inclusion of the transform Z_k and of its powers among themselves (for different values of k) and in relation to the arithmetic mean. An interesting result concerning the definitions Z_k is the following: If k contains h as a factor, then Z_k includes Z_h; if k is prime to h, then the only sequences evaluated by both Z_k and Z_h are the convergent sequences. Finally, a study is made of the reversibility of the transformation $\alpha I + (1 - \alpha)Z_k$, where I is the identical transformation, and α is a complex number.

2. P-matrices and N-matrices

DEFINITION 1. *The matrix* $A = (a_{n\nu})_{-\infty}^{\infty} = (a_{n-\nu})_{-\infty}^{\infty}$ *consists of numbers corresponding to all values of* n *and* k, *positive, negative and zero. There corresponds to it the Laurent series* $\sum_{-\infty}^{\infty} a_\nu z^\nu$ *and it will accordingly be called an L-matrix.*

[1] *W. A. Hurwitz:* A Trivial Tauberian Theorem, Bulletin Amer. Math. Soc. 32 (1926), 77-82. See also *R. P. Agnew*, On Deferred Cesàro Means, Ann. of Math., 33 (1932), 413-421, for a more general class of transforms.

[2] It is sometimes convenient to define $ky_0 = x_0$, $ky_1 = x_1$, $\cdots\cdots$, $ky_{k-2} = x_{k-2}$; we shall denote the transformation by Z_k in either case; clearly the values of x_0, x_1, $x_2 \cdots$, x_{k-2} are irrelevant.

To the product of two L-matrices A and B clearly corresponds the product of the two Laurent series:

$$\sum_{-\infty}^{\infty} a_\nu z^\nu \sum_{-\infty}^{\infty} b_\nu z^\nu = \sum_{-\infty}^{\infty} c_\nu z^\nu.$$

DEFINITION 2. *When the numbers n and k assume only positive values and zero, the matrix $(a_{n-\nu})_0^\infty$ will be called a T-matrix. First special case: $a_{-\nu} = \bar{a}_\nu$; the matrix becomes Hermitian and has a relation to Fourier Series. Second special case: $a_{-\nu} = 0$. The matrix becomes triangular.*

DEFINITION 3. *Corresponding to the triangular matrix ($a_{-\nu} = 0$), there is a power series $\sum_0^\infty a_\nu z^\nu = p(z)$. The matrix will be called a P-matrix. We shall use the notation $P = (a_0, a_1, a_2, \cdots) =$*

$$\begin{array}{|cccccc} a_0 & & & & & \\ a_1 & a_0 & & & & \\ a_2 & a_1 & a_0 & & & \\ a_3 & a_2 & a_1 & a_0 & & \\ - & - & - & - & - & - \\ - & - & - & - & - & - \end{array} \sim p(z).$$

DEFINITION 4. *Any matrix $(a_{n\nu})$ is of finite rank r if there exists an element $a_{n\nu} \neq 0$ for $|n - \nu| = r - 1$ and if all $a_{n\nu} = 0$, for $|n - \nu| \geqq r$. In particular, a P-matrix is of rank r when and only when $a_{r-1} \neq 0$, while $a_n = 0$ for $n \geqq r$.*

From now on we shall restrict ourselves to triangular matrices.

DEFINITION 5. *A diagonal matrix is one for which $a_{n\nu} = 0$ for $\nu \neq n$.*

DEFINITION 6. *The matrix with elements*

$$a_{n\nu} = \frac{\alpha_{n-\nu}}{A_n}, \quad \text{where} \quad A_n = \sum_0^n \alpha_\nu \neq 0,$$

will be called an N-matrix (Nörlund). We associate with this matrix the transform $y_n = \sum_{\nu=0}^n a_{n\nu} x_\nu$.

THEOREM 1. *An N-matrix may be written as the product of a diagonal matrix and a P-matrix. That is, $N = \delta P$, where δ has the diagonal elements A_n^{-1}.*

The proof is obvious. Clearly $\delta P = \left(\frac{1}{A_n}\alpha_{n-\nu}\right) = N$.

REMARK. The product $P\delta = \left(\frac{\alpha_{n-\nu}}{A_\nu}\right) \neq \delta P$. We shall say that the P-matrix and the N matrix are *corresponding*.

THEOREM 2. *The product of two P-matrices is a P-matrix. In fact, if $P \sim p(z)$ and $Q \sim q(z)$, then $PQ \sim p(z)q(z) = \sum_0^\infty (p, q)_n z^n$ where $(p, q)_n = \sum_0^n p_\nu q_{n-\nu}$.*

COROLLARY. The multiplication of two P-matrices is permutable.

THEOREM 3. *The product of two P-matrices of finite rank r and s respectively is of finite rank t, where $t = r + s - 1$.*

The proof is simple if we make use of the power series representation. In our case the power series reduce to polynomials.

THEOREM 4. *An N-matrix is of finite rank r if and only if the corresponding P-matrix is of rank r.*

The proof is obvious. The rank of a matrix is clearly unaltered when all the elements are multiplied by the elements of a diagonal matrix.

DEFINITION 7. *The symmetric product of two N-matrices N_1 and N_2 is $N_3 = ((\gamma_{n-\nu})/\Gamma_n)$, where $\gamma_n = (\alpha, \beta)_n$, and $\Gamma_n = (\gamma, 1)_n$. We assume $\Gamma_n \neq 0$. We write symbolically $N_3 = N_1 \circ N_2$.*

THEOREM 5. *For two N-matrices of finite rank r and s respectively, the ordinary product and the symmetric product are essentially identical; that is, for $n \geqq r + s - 2$ the elements of both products coincide.*

Let the matrix $(c_{n\nu})$ correspond to the ordinary product and $(\gamma_{n\nu})$ to the symmetric product, so that,

$$c_{n\nu} = \frac{1}{A_n} \sum_{\lambda=\nu}^{n} \frac{\alpha_{n-\lambda}\beta_{\lambda-\nu}}{B_\lambda}, \qquad \gamma_{n\nu} = \frac{(\alpha, \beta)_{n-\nu}}{(\alpha, \beta, 1)_n}.$$

Now by hypothesis $\alpha_{r-1} \neq 0$; $\alpha_\nu = 0$, $\nu \geqq r$; $\beta_{s-1} \neq 0$; $\beta_\nu = 0$, $\nu \geqq s$. Furthermore $B_n = B_{n-1} = \cdots = B_s = B_{s-1} = \beta_0 + \beta_1 + \cdots + \beta_{s-1}$, and $n - r + 1 \geqq s - 1$. Hence

$$c_{n\nu} = \frac{1}{A_n} \sum_{n-r+1}^{n} \frac{\alpha_{n-\lambda}\beta_{\lambda-\nu}}{B_\lambda} = \frac{(\alpha, \beta)_{n-\nu}}{A_n B_{s-1}}.$$

To prove the theorem it remains only to consider the denominators A_nB_{s-1} and $(\alpha, \beta, 1)_n$. We wish to prove

$$\alpha_0 B_n + \alpha_1 B_{n-1} + \cdots + \alpha_{r-1} B_{n-r+1} = B_{s-1}(\alpha_0 + \alpha_1 + \cdots + \alpha_{r-1}).$$

Now $B_\nu = B_{s-1}$ for $\nu \geqq n - r + 1$. Thus the two expressions are equal and the theorem is proved.

COROLLARY. All N-matrices of finite rank are consistent. For let N_1 and N_2 be two N-matrices of finite rank. Then N_1N_2 is essentially identical with $N_1 \circ N_2$ which is the same as $N_2 \circ N_1$, and hence essentially identical with N_2N_1. From this consistency follows.

NOTE. It is of course well-known that all N-matrices are consistent.

THEOREM 6. *If the triangular matrix $A = (a_{n\nu})$* is permutable with *one* P-matrix $(\beta_0, \beta_1, \beta_2, \cdots)$, $\beta_1 \neq 0$; then the matrix A is a P-matrix and is therefore permutable with *every* P-matrix.

Let $AP = (c_{n\nu})$, $PA = (\gamma_{n\nu})$; then

$$c_{n\nu} = \sum_{\lambda=\nu}^{n} a_{n\lambda}\beta_{\lambda-\nu}, \qquad \gamma_{n\nu} = \sum_{\lambda=\nu}^{n} \beta_{n-\lambda}a_{\lambda\nu}.$$

For the special value $\nu = n - 1$ we obtain, since $\beta_1 \neq 0$, $a_{nn} = a_{n-1,n-1}$, that is the elements in the principal diagonal are equal. Assuming now that the elements in each diagonal, counting from the principal diagonal, are equal, i.e., $a_{nn} = a_{hh}$, $a_{n,n-1} = a_{h+1,h}$, $\cdots$ $a_{n,n-\lambda} = a_{h+\lambda,h}$ we obtain from the assumption $c_{n,n-\lambda} = \gamma_{n,n-\lambda}$ the relation $a_{n,n-\lambda-1} = a_{h+\lambda+1,h}$ which says that the elements in the next following diagonal are equal. Hence the theorem is proved by induction.

3. Simple matrices and their powers

DEFINITION 8. *A simple matrix is an N-matrix of finite rank k whose corresponding P-matrix consists of elements which are zeros and unity. If all $\beta_n = 1$, $n < k$ and $\beta_n = 0$, $n \geqq k$, the matrix will be called a Z-matrix and will be denoted by Z_k.*

EXAMPLES.

P-matrix	Corresponding transformation
1. $k = 1$.	
$U_1 = (1, 0, 0, 0, \cdots)$;	Z_1: $y_n = x_n$.
2. $k = 2$.	
$U_2 = (1, 1, 0, 0, 0, \cdots)$;	Z_2: $2y_n = x_n + x_{n-1}$.
3. $k = 3$.	
$U_3 = (1, 1, 1, 0, 0, \cdots)$	Z_3: $3y_n = x_n + x_{n-1} + x_{n-2}$.
$S = (1, 0, 1, 0, 0, \cdots)$	$2y_n = x_n + x_{n-2}$.

THEOREM 7. *If the triangular matrix A is permutable with a simple matrix for which $\beta_1 = 1$, then A is a P-matrix.*

This is a corollary of Theorem 6.

THEOREM 8. *A necessary condition that the triangular matrix A be permutable with one Z_k is that A be a P-matrix; it is then permutable with every Z_k.*

This follows from the assumption $AZ_k = Z_kA$ and from Theorem 6.

THEOREM 9. *The transformation Z_k is regular for $k = 1, 2, 3, \cdots$ and is more powerful than convergence for $k = 2, 3, 4, \cdots$.*

The regularity is evident. That it is more powerful than convergence follows from the example, given below in the proof of Theorem 12, namely: $x_n = \omega^n$; where $\omega^k = 1$, $\omega \neq 1$.

THEOREM 10. *Necessary and sufficient conditions that the product of a diagonal matrix (δ_n) by a P-matrix $(\beta_0, \beta_1, \cdots)$ be regular are:*

$$\lim_{n\to\infty} \delta_n \beta_{n-\nu} = 0, \qquad \nu = 0, 1, 2, \cdots \tag{1}$$

$$\lim_{n\to\infty} \delta_n (\beta, 1)_n = 1 \tag{2}$$

$$\delta_n (|\beta|, 1)_n = 0(1). \tag{3}$$

The elements of the product transformation are clearly $c_{n\nu} = \delta_n \beta_{n-\nu}$; and the conditions of the theorem are exactly the Toeplitz conditions which are necessary and sufficient for the regularity of the product matrix.

COROLLARY 1. If $\dfrac{\delta_n}{\delta_{n-1}} \to 1$, then the condition (1) may be replaced by the simpler condition

$$\lim \delta_n \beta_n = 0. \tag{1'}$$

COROLLARY 2. If, in particular, $\delta_n = \dfrac{1}{n+1}$, the conditions become

$$\frac{\beta_n}{n} \to 0; \qquad \frac{(\beta, 1)_n}{n} \to 1; \qquad (|\beta|, 1)_n = 0(n).$$

THEOREM 11. *The arithmetic mean M is more powerful than any Z_k, $k = 1, 2, 3, \cdots$.*

PROOF. We have M: $(n+1)\, s_n = x_0 + x_1 + \cdots x_n$; Z_k : $kt_n = x_n + x_{n-1} + \cdots + x_{n-k+1}$. Define the P-matrices $U = (1, 1, 1, \cdots)$, all elements unity, and $U_k = kZ_k$; so that the elements in the first k diagonals of U_k (beginning with the principal diagonal) are *unity* and all the other elements *zero*. Also define the diagonal matrix δ by $\delta_n = \dfrac{1}{n+1}$. Then $M = \delta U$ and $MZ_k^{-1} = k\delta U U_k^{-1}$. If we now take $P = kUU_k^{-1}$, we see that $MZ_k^{-1} = \delta P$. In order to prove the theorem we must prove that MZ_k^{-1} is regular; or, what amounts to the same thing, that δP is regular. We shall make use of Corollary 2 to Theorem 10. We wish to determine the numbers β_n corresponding to the matrix P. For this purpose we use the representation by power series. We have

$$U_k \sim 1 + u + u^2 + \cdots + u^{k-1} = \frac{1-u^k}{1-u},$$

$$U_k^{-1} \sim \frac{1-u}{1-u^k}; \quad \text{hence} \quad P \sim \frac{k}{1-u^k}.$$

Thus $\beta_n = k$, if n is a multiple of k,
$= 0$, in every other case.

From this it immediately follows that the Toeplitz conditions are satisfied.

It remains to prove that $Z_k M^{-1}$ is not regular for any k, or to give an example of a series summable by M but not by Z_k. This is included in the second part of Theorem 12.

We now introduce the powers of $Z_k = \dfrac{1}{k} U_k$.

THEOREM 12. *The methods Z_k^λ and M are not comparable for k, $\lambda \geqq 2$.*

PROOF. a) Let $x_n = n\omega^n$, where $\omega^k = 1$, $\omega \neq 1$. Write

$$Z_k^2 \sim \frac{1}{k^2}\left(\frac{1-u^k}{1-u}\right)^2 = \sum a_\nu u^\nu;$$

then

$$t_n = \sum_{\nu=0}^{n} a_\nu x_{n-\nu}$$

is the nth coefficient of $(\sum a_\nu u^\nu)(\sum x_\nu u^\nu)$. But

$$\sum_0^\infty x_\nu u^\nu = \sum_0^\infty \nu(\omega u)^\nu = \frac{\omega u}{(1-\omega u)^2},$$

and

$$(\sum a_\nu u^\nu)(\sum x_\nu u^\nu) = k^{-2}\left(\frac{1-u^k}{1-u}\right)^2 \frac{\omega u}{(1-\omega u)^2},$$

which is a polynomial of degree $2k-3$; hence $t_n = 0$ for $n > 2k-3$, and $t_n \to 0$. Thus the sequence $\{x_n\}$ is Z_k^2-summable, whence Z_k^λ-summable, but clearly not M-summable.

b) For the converse let $x_n = \zeta^n$, where $|\zeta| = 1$, $\zeta \neq 1$. Now

$$Z_k^\lambda \sim \frac{1}{k^\lambda}\left(\frac{1-u^k}{1-u}\right)^\lambda = \sum_{\nu=0}^{\lambda(k-1)} b_\nu u^\nu,$$

and

$$t_n = \sum_0^n b_\nu \zeta^{n-\nu} = \frac{\zeta^n}{k^\lambda}\left(\frac{1-\zeta^{-k}}{1-\zeta^{-1}}\right)^\lambda \quad \text{for} \quad n \geqq \lambda(k-1),$$

Hence $\lim t_n$ exists if and only if $\zeta^k = 1$, while the sequence is M-summable if $|\zeta| = 1$, $(\zeta \neq 1)$. This proves the theorem.

THEOREM 13. *Let* $\sum_0^\infty |a_n - a_{n+1}| < \infty$; *the series* $\sum a_n z^n$ *is summable* Z_k *for* $z^k = 1$, $z \neq 1$.

PROOF. We have

$$x_n = \sum_0^n a_\nu z^\nu = a_n T_n + \sum_0^{n-1} T_\nu(a_\nu - a_{\nu+1}),$$

where $T_n = \dfrac{1-z^{n+1}}{1-z}$. Now

$$\lim_{n\to\infty} \sum_0^n T_\nu(a_\nu - a_{\nu+1}) = \sum_0^\infty T_\nu(a_\nu - a_{\nu+1})$$

exists; hence we need only to prove that the sequence $\{a_n T_n\}$ is summable. We first note that the series $\sum (a_n - a_{n+1})$ converges, hence $\lim a_n = \alpha$ exists. Put

$$ky_n = a_n T_n + \cdots + a_{n-k+1} T_{n-k+1} = \frac{1}{1-z} \sum_{n-k+1}^n a_\nu(1 - z^{\nu+1});$$

then clearly $\sum_{n-k+1}^n a_\nu \to k\alpha$. Finally

$$\sum_{n-k+1}^n a_\nu z^{\nu+1} = z^{n-k+2}(a_{n-k+1} + \cdots + a_n z^{k-1}) \to 0 \quad \text{as} \quad n \to \infty,$$

which proves the theorem.

Taking k even $= 2q$, $z = -1$, and a_n decreasing, the assumptions of the theorem are satisfied; hence $\sum (-1)^n a_n$ is summable Z_{2q}.

4. Comparison of Z_k and Z_l

DEFINITION 9. *If A and B are regular transformations, then the set of sequences summed by both A and B is called the intersection (the common part) of A and B*

and will be denoted by (A, B). *It is evident that* (A, B) *contains the domain of convergent sequences.*

THEOREM 14. *If* h *is a factor of* k, *then* Z_k *includes* Z_h. *Letting* $k = hd$, *we have:*

$$Z_h : hs_n = \sum_{n-h+1}^{n} x_\nu, \quad \text{and} \quad s_n \to \zeta.$$

$$Z_k : kt_n = \sum_{n-hd+1}^{n} x_\nu = \sum_{g=0}^{d-1} \sum_{\nu=n-h+1}^{n} x_{\nu-gh}$$

and

$$t_n = \frac{1}{d} \sum_{g=0}^{d-1} \frac{1}{h} \sum_{\nu=n-h+1}^{n} x_{\nu-gh}.$$

Since d and h are independent of n, we have

$$t_n \to \frac{1}{d} d\zeta = \zeta.$$

COROLLARY. If d is a divisor of both h and k, then Z_d is included in both Z_h and Z_k.

THEOREM 15. *If* h *and* k *have the greatest common divisor* g, *then* $(Z_h, Z_k) = Z_g$.

We first prove the

LEMMA. *Let* Z_h *and* Z_k *both sum the sequence* $\{x_n\}$, *where* $h > k$; *then* Z_{h-k} *also sums the sequence* $\{x_n\}$.

For we have:

$$Z_h : \quad hs_n = x_n + x_{n-1} + \cdots + x_{n-h+1} \to h\zeta,$$

$$Z_k : \quad kt_n = x_n + x_{n-1} + \cdots + x_{n-k+1} \to k\zeta.$$

Hence $hs_n - kt_n = x_{n-k} + \cdots + x_{n-h+1} \to (h - k)\zeta$.

Now let

$$u_n = \frac{hs_n - kt_n}{h - k}$$

so that

$$(h - k)u_n = hs_n - kt_n = x_{n-k} + x_{n-k+1} + \cdots + x_{n-h+1} \to (h - k)\zeta.$$

Now returning to the proof of the theorem, suppose first that a sequence is summable by Z_g, then by Theorem 14 it is summable by both Z_h and Z_k, and is accordingly included in both Z_h and Z_k. For the converse, choose integers α, β such that $\alpha h - \beta k = g$. Now if a sequence is summable by both Z_h and Z_k, then by Theorem 14 it is summable by $Z_{\alpha h}$ and $Z_{\beta k}$ and hence by the Lemma it is summable by $Z_{\alpha h - \beta k}$ which is Z_g.

COROLLARY. If h is prime to k, then $(Z_h, Z_k) = (I)$; that is, the only sequences evaluated by both transformations are the convergent sequences.

We now give a single example of a series summable Cesàro C_r for all $r > 0$

but not summable Z_k for any k. Consider the sequence $\{\zeta^n\}$, where $|\zeta| = 1$ and $\zeta^k \neq 1$ for all k. The nth C_r mean of this sequence is

$$\frac{1}{\binom{n+r}{n}} \sum_0^n \binom{\nu+r-1}{\nu} \zeta^\nu,$$

which evidently approaches zero; but the sequence is not summable Z_k. (In fact, it is not summable Z_k^λ; see the proof of Theorem 12.)

COROLLARY. The domains corresponding to the transformations Z_n increase to a limiting domain D which is smaller than the domain of C_r.

We now give two other interesting examples.

a) The sequence $x_n = (-1)^n(n/\log n)$ is summable Z_{2q} for every q. For $q = 1$,

$$\tfrac{1}{2}(x_n + x_{n-1}) = (-1)^n\left(\frac{n}{\log n} - \frac{n-1}{\log (n-1)}\right) = (-1)^n \Delta_n$$

and

$$\Delta_n = \int_{n-1}^n \frac{-1 + \log x}{(\log x)^2} dx < \int_{n-1}^n \frac{dx}{\log x} < \frac{1}{n-1} \to 0.$$

Clearly then the sequence is also summable Z_{2q} and summable M; but it is not summable C_r for any r less than unity, because the sequence $\{n^{-r}x_n\}$ is not bounded.

b) Form the sequence $\{y_n\}$ from the sequence $\{x_n\}$ as follows: for a given g take the terms of the sequence $\{x_n\}$ in groups of $2g$ terms, and interpolate after each group the element zero. Clearly this new sequence $\{y_n\}$ is summable Z_{2g+1} (hence M-summable), but not summable C_r for any r less than unity. There is no such single example for *all* g, as is seen from the corollary to Theorem 15.

5. Absolute summability Z_k

A sequence $\{x_n\}$ is called absolutely summable by the transform $t_n = \sum_{\nu=0}^n a_{n\nu}x_\nu$ if $\sum_1^\infty |t_n - t_{n-1}| < \infty$. In particular, absolute summability Z_k is defined by $\sum_1^\infty |y_n - y_{n-1}| < \infty$, where $ky_n = \sum_{\nu=0}^{k-1} x_{n-\nu}$, and $k(y_n - y_{n-1}) = x_n - x_{n-k}$.

THEOREM 16. *An alternating series of decreasing terms is absolutely summable Z_{2g}; it is absolutely summable Z_{2g+1} if and only if it is absolutely convergent.*

To prove the first part of the theorem we have to show that $\sum_n |x_n - x_{n-2g}| < \infty$. Now $x_n - x_{n-2g} = \sum_{\nu=0}^{g-1}(x_{n-2\nu} - x_{n-2\nu-2})$, and $x_n = \sum_0^n (-1)^\nu u_\nu$, $x_n - x_{n-2} = (-1)^n(u_n - u_{n-1})$; but from our assumption

$$\sum_2^n |x_\nu - x_{\nu-2}| = \sum_2^n (u_{\nu-1} - u_\nu) = u_1 - u_n < u_1,$$

which proves the first part of our theorem.

To prove the second part, we write

$$(2g+1)(y_n - y_{n-1}) = x_n - x_{n-(2g+1)} = x_n - x_{n-2g} + x_{n-2g} - x_{n-(2g+1)}$$
$$= x_n - x_{n-2g} + u_{n-2g}(-1)^{n-2g}.$$

But from the previous theorem $\sum_n |x_n - x_{n-2g}| < \infty$, hence $\sum_n |y_n - y_{n-1}| < \infty$ if and only if $\sum |u_n| < \infty$.

REMARK. The series $\sum_0^\infty (-1)^n$ is absolutely summable Z_{2g} but not absolutely M-summable. We have

$$x_n = \tfrac{1}{2}\{1 + (-1)^n\}, \qquad x_n - x_{n-2} = 0.$$

But

$$2t_n = \frac{2}{n+1}\sum_0^n x_r = \frac{1}{n+1}\left\{n+1+\sum_0^n (-1)^r\right\} = 1 + \frac{1+(-1)^n}{2(n+1)},$$

$$2(t_n - t_{n-1}) = \frac{1}{2n(n+1)} + (-1)^n \frac{1}{2}\left(\frac{1}{n} + \frac{1}{n+1}\right);$$

hence $\sum |t_n - t_{n-1}| = \infty$.

6. The inverse of the transform $A = \alpha I + (1-\alpha)Z_k$; $k \geqq 2$

We shall now consider the transformation $A = \alpha I + (1-\alpha)Z_k$, where I is the identical transformation and Z_k is ($k \geqq 2$):

$$x'_n = \frac{x_n + x_{n-1} + \cdots + x_{n-k+1}}{k}, \qquad n \geqq k-1;$$

that is, the transformation A is:

$$y_n = \alpha x_n + (1-\alpha)\frac{x_n + x_{n-1} + \cdots + x_{n-k+1}}{k}.$$

This transformation is clearly regular. We ask the question, for what complex values of α is it reversible?

We have

$$y_n = \left(\alpha + \frac{1-\alpha}{k}\right)x_n + \frac{1-\alpha}{k}(x_{n-1} + \cdots + x_{n-k+1});$$

write

$$\frac{k}{1+(k-1)\alpha}y_n = z_n = x_n + \beta(x_{n-1} + \cdots + x_{n-k+1}),$$

$$\beta = \frac{1-\alpha}{1+(k-1)\alpha}, \qquad 1 + (k-1)\alpha \neq 0.$$

Let all $z_n = 0$, then x_n must approach zero. To solve for x_n put $x_n = r^n$, then we obtain for r the equation

$$0 = r^n + \beta(r^{n-1} + \cdots + r^{n-k+1}).$$

Let $r \neq 0$, then r must be a root of the equation

$$r^{k-1} + \beta(r^{k-2} + \cdots + r + 1) = 0.$$

In order that $x_n \to 0$ we must have $r^n \to 0$. Hence the absolute value of each root r must be less than unity; that is, each root must lie in the unit circle. Call the roots $r_1, r_2, \cdots, r_{k-1}$. Now as to sufficiency. We take $x_\nu = y_\nu = 0$, $\nu = 0, 1, \cdots, k-2$. Then

$$\begin{aligned}\sum_{k-1}^{\infty} z_n u^n &= \sum_{k-1}^{\infty} x_n u^n + \beta \sum_{k-1}^{\infty} (x_{n-1} + \cdots + x_{n-k+1})u^n \\ &= \sum_{0}^{\infty} x_n u^n (1 + \beta u + \cdots + \beta u^{k-1}) \\ &= \sum_{0}^{\infty} x_n u^n \prod_{1}^{k-1} (1 - r_\nu u).\end{aligned}$$

Thus

$$\sum x_n u^n = \sum z_n u^n \prod_{1}^{k-1} \frac{1}{1 - r_\nu u} = (\sum z_n u^n)(\sum p_n u^n);$$

$\sum p_n u^n = \prod_{\nu=1}^{k-1} \sum_{n=0}^{\infty} r_\nu^n u^n$ *is dominated by* $(\sum \rho^n u^n)^{k-1}$, *where* $\rho = \max |r_\nu|$.

Hence $|p_n| \leqq (-1)^n \binom{1-k}{n} \rho^n$, and thus $p_n \to 0$ while $\sum_0^\infty |p_n| < \infty$, $\sum_0^\infty p_n = \prod_1^{k-1} (1 - r_\nu)^{-1} = \dfrac{1}{1 + (k-1)\beta} = \dfrac{1 + (k-1)\alpha}{k}$, which are precisely the Toeplitz conditions for this case.

REMARK. For $\alpha = \dfrac{1}{1-k}$, $k > 1$, the transformation reduces to Z_{k-1}, which is not reversible (Theorem 9), except for $k = 2$. We have thus proved:

THEOREM 17. *The transformation* $\alpha I + (1 - \alpha)Z_k$, $k \geqq 2$, *is reversible if and only if all roots of the equation*

$$[1 + (k-1)\alpha]r^{k-1} + (1 - \alpha)\frac{1 - r^{k-1}}{1 - r} = 0$$

lie inside the unit circle. For $k = 2$ *this condition becomes*

$$\left|\frac{1 - \alpha}{1 + \alpha}\right| < 1, \quad \text{or} \quad R\alpha > -1.$$

In the singular case; $k = 2$, $\alpha = -1$, *the transformation reduces to the reversible one* $y_n = x_{n-1}$.

7. Multiplication of summable series

In this section we consider the question of the summability of the product of two series each of which is Nörlund-summable. We have the following result:

THEOREM 18. *If $\sum_0^\infty u_n$ and $\sum_0^\infty v_n$ are Nörlund-summable P and Q respectively, if $p_n \geqq 0$, $q_n \geqq 0$, and if one of these summabilities is absolute, then the Cauchy-product of the two series is Nörlund-summable by $R = P \circ Q$.*

Let

$$U_n = (u, 1)_n, \qquad V_n = (v, 1)_n, \qquad w_n = (u, v)_n, \qquad W_n = (u, V)_n = (U, v)_n,$$

$$P_n\eta_n = (p, U)_n, \qquad Q_n\varphi_n = (q, V)_n, \; r_n = (p, q)_n, \qquad R_n = (P, q)_n = (p, Q)_n.$$

By hypothesis, $\varphi_n \to \varphi$, $\eta_n = \eta$, $\sum |\eta_n - \eta_{n-1}| < \infty$. We have $P_n\eta_n - P_{n-1}\eta_{n-1} = (p, u)_n$, where $\eta_{-1} = P_{-1} = 0$. Now

$$(r, W)_n = (p, q, u, V)_n = \sum_0^n (p, u)_{n-\nu}(q, V)_\nu,$$

$$= \sum_0^n (P_{n-\nu}\eta_{n-\nu} - P_{n-\nu-1}\eta_{n-\nu-1})Q_\nu\varphi_\nu$$

$$= \sum_0^n [P_{n-\nu}(\eta_{n-\nu} - \eta_{n-\nu-1}) + p_{n-\nu}\eta_{n-\nu-1}]Q_\nu\varphi_\nu.$$

Suppose first $\varphi = 0$, $\eta = 0$; then

$$\left|\sum_0^n (\eta_{n-\nu} - \eta_{n-\nu-1})P_{n-\nu}Q_\nu\varphi_\nu\right| \leqq (p, Q)_n \sum_0^n |\eta_{n-\nu} - \eta_{n-\nu-1}||\varphi_\nu|,$$

and

$$\sum_0^n |\eta_{n-\nu} - \eta_{n-\nu-1}||\varphi_\nu| \leqq \sum_{\nu \leqq n/2} + \sum_{\nu > n/2} = o(1) + o(1).$$

Furthermore

$$\left|\sum_0^n p_{n-\nu}\eta_{n-\nu-1}Q_\nu\varphi_\nu\right| \leqq (p, Q)_n \sum_0^n |\eta_{n-\nu-1}\varphi_\nu|,$$

and

$$\sum_0^n |\eta_{n-\nu-1}\varphi_\nu| = \sum_{\nu \leqq n/2} + \sum_{\nu > n/2} = o(1) + o(1).$$

Hence

$$\frac{(r, W)_n}{R_n} = \frac{(r, W)_n}{(p, Q)_n} \to 0 \quad \text{as} \quad n \to \infty.$$

For the general case we need only apply our result to the sequences $\{U_n - \eta\}$, $\{V_n - \varphi\}$. This proves the theorem. Since Z_k is a Nörlund matrix with non-negative elements, we have the

COROLLARY: If Z_k and Z_l evaluate the series $\sum u_n$ and $\sum v_n$ respectively to the sums U and V, one of the summabilities being absolute, then the symmetric product of Z_k and Z_l evaluates the Cauchy product of the series to UV.

DARTMOUTH COLLEGE
UNIVERSITY OF CINCINNATI

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EXTENSIONS OF DIFFERENTIAL FIELDS, II

BY E. R. KOLCHIN

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Introduction

We consider two differential fields $\mathcal{F}$ and $\mathcal{G}$ of characteristic 0 such that $\mathcal{F} \subset \mathcal{G}$, and ask how many elements of $\mathcal{G}$ it is necessary to adjoin to $\mathcal{F}$ in order to obtain $\mathcal{G}$. The first answer to a question of this kind was furnished by J. F. Ritt,[1] who proved the following analog to an algebraic theorem of Luroth's: If $\mathcal{F}$ is an *ordinary*[2] differential field, and if $\mathcal{G} = \mathcal{F}\langle \alpha_1, \alpha_2 \rangle \subseteq \mathcal{F}\langle y \rangle$, where y is an unknown, then $\mathcal{G}$ is a simple extension of $\mathcal{F}$.[3] Ritt's proof applied only to differential fields of functions meromorphic in a region of the complex plane, as it made use of function-theoretic considerations. The second result along these lines, given by the author,[4] states that if $\mathcal{F}$ is a differential field of characteristic 0 with m types of differentiation $\delta_1, \cdots, \delta_m$, if $\mathcal{F}$ contains m elements $\xi_1, \cdots, \xi_m$ whose Jacobian $|\delta_i \xi_j|$ does not vanish, if $\mathcal{G}$ is differentially algebraic over $\mathcal{F}$,[5] and if $\mathcal{G}$ is a *finite* extension of $\mathcal{F}$, then $\mathcal{G}$ is already a *simple* extension of $\mathcal{F}$.

From this result it easily follows that if the degree of differential transcendency of $\mathcal{G}$ over $\mathcal{F}$ is $d \geqq 0$, and if $\mathcal{G}$ is a finite extension of $\mathcal{F}$, then $\mathcal{G}$ is a $(d+1)$-fold extension of $\mathcal{F}$. For example, if $\mathcal{F} \subset \mathcal{G} \subseteq \mathcal{F}\langle y \rangle$, and if $\mathcal{G}$ is a finite extension of $\mathcal{F}$, then $\mathcal{G}$ is a two-fold extension of $\mathcal{F}$. When $\mathcal{F}$ is ordinary, Ritt's theorem shows that $\mathcal{G}$ is then a simple extension of $\mathcal{F}$.

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[1] RITT, *Differential equations from the algebraic standpoint*, American Mathematical Society Colloquium Publications, vol. XIV, New York, 1932. See chapter VIII.

[2] A differential field with m types of differentiation is called *ordinary* or *partial* according as $m = 1$, or $m > 1$.

[3] $\mathcal{F}\langle \cdots \rangle$ indicates differential field extension; $\mathcal{F}\{\cdots\}$ indicates differential ring extension.

[4] KOLCHIN, *Extensions of differential fields*, I, Annals of Mathematics, vol. 43 (1942), pp. 724–729.

[5] A differential field $\mathcal{G}$ will be called *differentially algebraic* over a differential subfield $\mathcal{F}$ if every element of $\mathcal{G}$ annuls a nonzero differential polynomial with coefficients in $\mathcal{F}$. If $\mathcal{G}$ is not differentially algebraic, it will be called *differentially transcendental*. Similarly, a single element of $\mathcal{G}$ will be spoken of as differentially algebraic or differentially transcendental over $\mathcal{F}$. We prefer these terms to "algebraically transcendental" and "hypertranscendental" (or "transcendentally transcendental") which have appeared in the literature, as more expressive. A set of elements of $\mathcal{G}$ will be called *differentially algebraically independent* over $\mathcal{F}$ if no finite subset of these elements annuls a nonzero differential polynomial with coefficients in $\mathcal{F}$. A set $\mathfrak{M}$ of differentially algebraically independent elements of $\mathcal{G}$ such that $\mathcal{G}$ is differentially algebraic over $\mathcal{F}\langle \mathfrak{M} \rangle$ will be called a *differential transcendency base* of $\mathcal{G}$ over $\mathcal{F}$. The cardinal number of a differential transcendency base of $\mathcal{G}$ over $\mathcal{F}$ will be called the *degree of differential transcendency* of $\mathcal{G}$ over $\mathcal{F}$. This number is unique. For the proof of this, and for a general discussion of these concepts, the reader is referred to two papers by H. W. RAUDENBUSH: Annals of Mathematics, vol. 34 (1933), pp. 509–517; Bulletin of the American Mathematical Society, vol. 40 (1934), pp. 714–720.

The main object of this note is to prove anew Ritt's theorem. The present proof (§1) is valid for any ordinary differential field $\mathcal{F}$ of characteristic 0, as no non-algebraic notions are employed. Also, the present statement of the theorem is slightly stronger than Ritt's, as it is not assumed here that $\mathcal{G}$ is a finite extension of $\mathcal{F}$.

It is perhaps not uninteresting that the theorem in question ceases to hold for partial differential fields. Examples are produced (§2) which show that for a partial differential field $\mathcal{F}$ one may find a differential field $\mathcal{G}$ between $\mathcal{F}$ and $\mathcal{F}\langle y\rangle$ which is a two-fold extension of $\mathcal{F}$ without being a simple extension, or which is not a finite extension at all.

1. The principal result

THEOREM. *Let $\mathcal{F}$ be an ordinary differential field of characteristic 0, let y be an unknown, and let $\mathcal{G}$ be a differential field such that $\mathcal{F} \subset \mathcal{G} \subseteq \mathcal{F}\langle y\rangle$. Then there exists an element ω in $\mathcal{G}$ such that $\mathcal{G} = \mathcal{F}\langle\omega\rangle$. If ω_1 and ω_2 are two such elements ω, then $\omega_2 = (a\omega_1 + b)/(c\omega_1 + d)$, where $a, b, c, d \in \mathcal{F}$.*

PROOF. Let Σ be the set of all differential polynomials in $\mathcal{G}\{z\}$ which vanish when $z = y$. Then Σ is a prime differential ideal the manifold of which is the general solution of some irreducible $A(z)$ in $\mathcal{G}\{z\}$. The coefficients in $A(z)$ are quotients of differential polynomials in $\mathcal{F}\{y\}$. We assume without loss of generality that one of these coefficients is unity.

Denote the least common denominator of these coefficients by $D(y)$, and let $B(y, z) = (D(y)A(z)$. Then $B(y, z) \in \mathcal{F}\{y, z\}$, and $B(y, y) = 0$. Let the orders of $B(y, z)$ in y and in z be p and q, respectively, and consider the separant $S(y, z) = \partial B(y, z)/\partial y_p$, where y_p is the p-th derivative of y.

We shall show that $S(y, y) \not\equiv 0$. Indeed, if $S(y, y)$ vanished, then $\partial A(z)/\partial y_p$, a differential polynomial in $\mathcal{G}\{z\}$ of order not exceeding q, would vanish when $z = y$, so that $\partial A(z)/\partial y_p$, would be divisible by $A(z)$. But one of the coefficients in $A(z)$ is unity, so that the corresponding coefficient in $\partial A(z)/\partial y_p$ is zero. Thus $\partial A(z)/\partial y_p$ could not be divisible by $A(z)$ unless $\partial A(z)/\partial y_p$ were zero, which would be absurd.

We now prove that $p \leqq q$. Indeed, suppose that $p > q$, and write $B(y, z)$ as a polynomial in y_p:

$B(y, z) = I_0(y, z) + I_1(y, z)y_p + \cdots + I_n(y, z)y_p^n$. Since $B(y, y) = 0$, and each $I_i(y, y)$ is free of y_p, it follows that each $I_i(y, y) = 0$. This implies that $S(y, y) = 0$. This contradiction proves that $p \leqq q$.

Let $\omega = H(y)/K(y)$ be any coefficient in $A(z)$ which effectively involves y, with $H(y)$ and $K(y)$ free of common divisor. By the preceding, the orders of $H(y)$ and $K(y)$ are less than or equal to q. Consequently $\omega K(z) - H(z)$, which is in $\mathcal{G}\{z\}$, and which vanishes when $z = y$, is divisible by $A(z)$:

$$\omega K(z) - H(z) = \alpha A(z), \qquad \alpha \in \mathcal{G}.$$

We shall show that $\mathcal{G} = \mathcal{F}\langle\omega\rangle$. To this end let $\theta = M(y)/N(y)$ be any element of $\mathcal{G}$. Now, $\theta N(z) - M(z)$ is in $\mathcal{G}\{z\}$ and vanishes when $z = y$. Hence

$\theta N(z) - M(z) \in \Sigma$. If $\bar{y}$ is an element of some extension of $\mathcal{F}$ such that $y \to \bar{y}$ generates a relative isomorphism of $\mathcal{F}\langle y\rangle$ with respect to $\mathcal{F}\langle\omega\rangle$,[6] then $z = \bar{y}$ is a non-singular solution of $\omega K(z) - H(z)$, so that $\theta N(\bar{y}) - M(\bar{y}) = 0$. Thus, under any isomorphism of $\mathcal{F}\langle y\rangle$ relative to $\mathcal{F}\langle\omega\rangle$, θ remains invariant. Hence $\theta \in \mathcal{F}\langle\omega\rangle$, and $\mathcal{G} = \mathcal{F}\langle\omega\rangle$.[7]

Now let ω_1 and ω_2 be two elements in $\mathcal{F}\langle y\rangle$ such that $\mathcal{F}\langle\omega_1\rangle = \mathcal{F}\langle\omega_2\rangle$. We have $\omega_2 = P(\omega_1)/Q(\omega_1)$, and $\omega_1 = R(\omega_2)/S(\omega_2)$, where $P(y)$, $Q(y)$, $R(y)$, $S(y) \in \mathcal{F}\{y\}$. If $z = \omega_1'$ is any solution of $\omega_2 Q(z) - P(z)$ which does not annul $P(z)Q(z)$, then $\omega_1 \to \omega_1'$ generates an isomorphism of $\mathcal{F}\langle\omega_1\rangle$ relative to $\mathcal{F}$ under which

$$\omega_2 = P(\omega_1)/Q(\omega_1) \to P(\omega_1')/Q(\omega_1') = \omega_2\,,$$

$$\omega_1 = R(\omega_2)/S(\omega_2) \to R(\omega_2)/S(\omega_2) = \omega_1\,,$$

so that $\omega_1' = \omega_1$. Thus $\omega_2 Q(z) - P(z)$ has only one solution, and therefore is of order 0 and of degree 1, so that $\omega_2 = (a\omega_1 + b)/(c\omega_1 + d)$.

2. Examples for partial differential fields

Let $\mathcal{F}$ be a differential field of characteristic 0 with two types of differentiation δ_1 and δ_2. Let y be an unknown.

EXAMPLE 1. *Let* $\mathcal{G} = \mathcal{F}\langle\delta_1 y, \delta_2 y\rangle$. *Then* $\mathcal{G}$ *is not a simple extension of* $\mathcal{F}$.

PROOF. Suppose that ω is an element of $\mathcal{G}$ for which $\mathcal{G} = \mathcal{F}\langle\omega\rangle$. Then we can write $\omega = P(y)/Q(y)$, where $P(y)$ and $Q(y)$ are in $\mathcal{F}\{y\}$, and have no common divisor. Since $\omega \in \mathcal{F}\langle\delta_1 y, \delta_2 y\rangle$ it is clear that y can not occur undifferentiated in $P(y)$ or $Q(y)$.

Let $P(y)/Q(y)$ be of order p, and let k be the highest value of j such that $\delta_1^j\delta_2^{p-j}y$ effectively appears in $P(y)/Q(y)$.

Since $\delta_1 y \in \mathcal{F}\langle\omega\rangle$ we have $\delta_1 y = M(\omega)/N(\omega)$, where $M(\omega)$, $N(\omega) \in \mathcal{F}\{\omega\}$. Let $M(\omega)/N(\omega)$ be of order q in ω, and let h be the highest value of j such that $\delta_1^j\delta_2^{q-j}\omega$ effectively appears in $M(\omega)/N(\omega)$.

Now, $\delta_1^h\delta_2^{q-h}\omega = \delta_1^h\delta_2^{q-h}P(y)/Q(y) = [U(y)\delta_1^{h+k}\delta_2^{p+q-h-k}y + V(y)]/Q(y)^{q+1}$, where $U(y)$, $V(y)$ are of order not exceeding $p + q$ and do not contain $\delta_1^{h+k}\delta_2^{p+q-h-k}y$. Furthermore, no derivative $\delta_1^a\delta_2^b\omega$ other than $\delta_1^h\delta_2^{q-h}\omega$ appearing in $M(\omega)/N(\omega)$ effectively involves $\delta_1^{h+k}\delta_2^{p+q-h-k}y$ when expressed in terms of y. Since $\delta_1 y = M(\omega)/N(\omega)$, this implies that $h + k = 1$, $p + q - h - k = 0$. But $p \geqq 1$, so that $p = 1$, $q = 0$, $h = 0$, and $\delta_1 y$ is a rational function of ω with coefficients in $\mathcal{F}$. Similarly for $\delta_2 y$, so that $\delta_1 y$ and $\delta_2 y$ are algebraically dependent over $\mathcal{F}$, which is a contradiction.

EXAMPLE 2. *Let* $\mathcal{G} = \mathcal{F}\langle\delta_1 y \cdot \delta_2 y, \delta_1^2 y \cdot \delta_2^2 y, \cdots, \delta_1^n y \cdot \delta_2^n y, \cdots\rangle$. *Then* $\mathcal{G}$ *is not a finite extension of* $\mathcal{F}$.

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[6] That is, an isomorphism of $\mathcal{F}\langle y\rangle$ which leaves invariant each element of $\mathcal{F}\langle\omega\rangle$. See KOLCHIN, loc. cit., p. 726.

[7] KOLCHIN, loc. cit., p. 726.

PROOF. Assume the contrary. Then for some n we may write

$$\delta_1^n y \cdot \delta_2^n y \in \mathcal{F}\langle \delta_1 y \cdot \delta_2 y, \cdots, \delta_1^{n-1} y \cdot \delta_2^{n-1} y \rangle.$$

That is, there exists a differential rational function $f(u, \cdots, u_{n-1})$ in the unknowns $u, \cdots, u_{n-1}$ with coefficients in $\mathcal{F}$ such that $\delta_1^n y \cdot \delta_2^n y = f(\delta_1 y \cdot \delta_2 y, \cdots, \delta_1^{n-1} y \cdot \delta_2^{n-1} y)$. In order that $f(\delta_1 y \cdot \delta_2 y, \cdots, \delta_1^{n-1} y \cdot \delta_2^{n-1} y)$ yield $\delta_1^n y \cdot \delta_2^n y$, $f(u, \cdots, u_{n-1})$ must effectively contain some $\delta_1^j \delta_2^j u_{n-j}$. Let k denote the greatest value of j such that $\delta_1^j \delta_2^j u_{n-j}$ is effectively contained in $f(u, \cdots, u_{n-1})$. Then $f(\delta_1 y \cdot \delta_2 y, \cdots, \delta_1^{n-1} y \cdot \delta_2^{n-1} y)$ effectively contains $\delta_1^n \delta_2^k y \cdot \delta_2^{n-k} y$, a contradiction.

OFFICE OF THE CHIEF OF ORDNANCE,
WASHINGTON, D. C.

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NOTE ON THE EXISTENCE OF SCALAR PRODUCTS IN NORMED LINEAR SPACES

BY FREDERICK A. FICKEN

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Let $\sum$ be a normed linear space. $\sum$ may or may not have the property that vectors P and Q define a unique scalar (P, Q), called their scalar product, which meets in $\sum$ some of the needs met in ordinary vector analysis by the scalar product $P \cdot Q$. If $\sum$ is complete and has scalar products, it is essentially Euclidean (cf. infra). A necessary and sufficient condition on the norm in order that (P, Q) be definable in an appropriate way has been given by Jordan and von Neumann.[1] Their main requirement is that

$$|P + Q|^2 + |P - Q|^2 = 2|P|^2 + 2|Q|^2$$

shall hold for any vectors P and Q of $\sum$.

The purpose of the present note is to show that the same object is gained by requiring, whenever a and b are real scalars and $|P| = |Q|$, that

$$|aP + bQ| = |bP + aQ|.$$

In the plane of P and Q, with $|P| = |Q|$, the mapping $aP + bQ \to bP + aQ$ amounts to a reflection about the line $t(P + Q)$, $-\infty < t < +\infty$. Our condition says that, for any P and Q, this mapping is isometric. Our result thus establishes this sort of symmetry about any axis as a criterion for the Euclidean character of a normed linear space.

$\sum$ is a normed linear space with vectors or points $0, P, Q, \cdots$, and real scalars $a, b, c, \cdots$. A non-negative real valued function $|X|$, the norm of X, has been defined so as to have the usual properties (e.g. properties 21, 22, 23 of $||f||$ in Reference 1). We are interested in defining a bilinear scalar product (X, Y) such that $|X|^2 = (X, X)$.

If this has been accomplished, and if $|P| = |Q|$, then

$$\begin{aligned} |aP + bQ|^2 &= (aP + bQ, aP + bQ) \\ &= a^2|P|^2 + 2ab(P, Q) + b^2|Q|^2 \\ &= (bP + aQ, bP + aQ) \\ &= |bP + aQ|^2 \end{aligned}$$

for any a and b. This proves the necessity of the condition stated in

THEOREM 1: *In order that a normed linear space with real scalars may permit the definition of a scalar product, it is necessary and sufficient that, whenever* $|P| = |Q|$, *and* a *and* b *are any scalars,*

N4 $$|aP + bQ| = |bP + aQ|.$$

[1] These Annals, Vol. 36 (1935) pp. 719–723.

To prove the sufficiency of this condition, we must adjoin N4 to the requirements on $|X|$ and present a definition of (X, Y). Jordan and von Neumann have constructed the required definition[2] under the (necessary and sufficient) condition that

N4′ $$|P+Q|^2+|P-Q|^2=2|P|^2+2|Q|^2.$$

Thus we need merely prove N4′ from N4.

The desired result will follow readily from the corollaries of

LEMMA 1. *If* $|P|=|Q|=\left|\frac{P+Q}{2}\right|$, *then* $P=Q$.

For, from $|P+Q|=2|Q|$, on using N4, it follows that

$$|P+Q-2(2Q)|=|2(P+Q)-2Q|=2|P|,$$
$$|P-Q-2Q|=2|P|.$$

Thus, when $n=0$,

(1) $$|(2n+1)(P-Q)-2Q|=2|P|=2|Q|.$$

It follows from (1) and N4 that

$$|2\{(2n+1)(P-Q)-2Q\}+(2n+3)(2Q)|$$
$$=|(2n+3)\{(2n+1)(P-Q)-2Q\}+2(2Q)|,$$

or

$$|(2n+3)(P-Q)-2Q|=2|P|.$$

By induction, then, (1) holds for $n=0, 1, 2, \cdots$.

But, by the triangle inequality, (1) yields

$$2|P|\geqq(2n+1)|P-Q|-2|Q|,$$

or

$$|P-Q|\leqq\frac{4|Q|}{2n+1}\qquad n=0, 1, 2, \cdots.$$

This is a contradiction unless $P=Q$, proving the lemma.

COROLLARY 1: *Unless* $|Q|P=|P|Q$, $|P+Q|<|P|+|Q|$.

Arrange the notation so that $|P|\leqq|Q|$. Then

$$|P+Q|=\left|P+|P|\frac{Q}{|Q|}+(|Q|-|P|)\frac{Q}{|Q|}\right|$$
$$\leqq\left|P+|P|\frac{Q}{|Q|}\right|+||Q|-|P||.$$

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[2] Specifically, in a complex space,

$$(P, Q)=\tfrac{1}{4}(|P+Q|^2-|P-Q|^2-i|iP+Q|^2+i|iP-Q|^2).$$

In a real space, take the real part. They obtain greater generality by discarding the assumption $|aP|=|a||P|$ in favor of the assumptions that $|iP|=|P|$ and that $|a|\to 0$ implies $|aP|\to 0$ for fixed P.

By the lemma, unless $|Q|P = |P|Q$, the first term on the right is *less* than $2|P|$, giving

$$|P+Q| < 2|P| + |Q| - |P| = |P| + |Q|.$$

COROLLARY 2: *For any U, V, and positive real number r, there are not more than two distinct real values of t such that $|U + t(V - U)| = r$.*

This is trivial unless $U \neq 0$, $V \neq 0$, and U and V are independent. If the assertion is false, we can find, among the vectors $U + t(V - U)$, independent vectors P and Q and a third vector $R = sQ + (1 - s)P$, $0 < s < 1$, such that $|P| = |Q| = |R| = r$. Since P and Q are independent, $|sQ|P \neq |(1 - s)P|Q$. By Corollary 1, then, $|R| < |sQ| + |(1 - s)P| = |P|$. This contradiction proves Corollary 2.[3]

Turning now to the proof of N4′, we notice that it is trivial unless $P \neq 0$, $Q \neq 0$, and P and Q are independent, so we assume these conditions met. It follows from N4, for any vectors P and Q, that

$$|P+Q| = |P - Q + 2Q| = \left|\frac{2|Q|}{|P-Q|}(P-Q) + \frac{|P-Q|}{|Q|}Q\right|$$

$$= \left|\frac{2|Q|}{|P-Q|}P + \frac{|P-Q|^2 - 2|Q|^2}{|P-Q||Q|}Q\right|,$$

or

$$(2)\quad |Q||P+Q||P-Q| = |2|Q|^2P + (|P-Q|^2 - 2|Q|^2)Q|.$$

Treating $|Q - P + 2P|$, $|P + Q - 2Q|$, and $|P + Q - 2P|$ similarly, we find, in turn,

$$|Q||P+Q||P-Q| = |2|Q|^2P + (|P-Q|^2 - 2|P|^2)Q|$$

$$= |2|Q|^2P + (2|Q|^2 - |P+Q|^2)Q|$$

$$= |2|Q|^2P + (2|P|^2 - |P+Q|^2)Q|.$$

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[3] One might say, accordingly, that spheres are strongly convex, in the sense that, if A and B are on a sphere, and C is *interior* to the segment joining them, then C is *interior* to the sphere. This situation in a complex space requires comment. If scalars are real, the segment joining A and B may be defined as the set of points $P = A + t(B - A)$ for $0 \leqq t \leqq 1$. Since a normed linear space is a metric space with distance PQ defined by $|P - Q|$, the segment may also be defined as the set of those points C such that $AC + CB = AB$. If complex scalars are admitted, the inequality restricting t becomes meaningless. If we desire to preserve the metric property, we wish the segment to consist of those points $P = tB + (1 - t)A$, with t complex, such that $|A - P| + |P - B| = |A - B|$. This entails $|t| + |1 - t| = 1$, whence t must again be real and $0 \leqq t \leqq 1$. Thus the segment joining A and B, when scalars are complex, is related to the (complex) 1-space $A + z(B - A)$ as the interval $0 \leqq x \leqq 1$ on the real axis is related to the complex plane. With this understanding, it follows in either case from Corollary 2 that spheres in Σ are strongly convex.

Applying Corollary 2, with $U = 2|Q|^2P$, $V - U = Q$, and $r = |Q||P + Q||P - Q|$, we find that, among the four numbers

$$a = |P - Q|^2 - 2|Q|^2 \qquad c = 2|Q|^2 - |P + Q|^2$$
$$b = |P - Q|^2 - 2|P|^2 \qquad d = 2|P|^2 - |P + Q|^2,$$

there are at most two distinct numbers. We have N4′ at once if either $a = d$ or $b = c$, so suppose $a \neq d$, $b \neq c$. If $a = c$ and $b = d$, then $|P + Q|^2 + |P - Q|^2 = 4|Q|^2 = 4|P|^2$, giving N4′ again. The only possible failure of N4′ would thus involve $a = b$ and $c = d$, whence $|P| = |Q|$.

Unless $|P + Q| = |P - Q|$, then, we may apply N4′ to $P + Q$ and $P - Q$, again obtaining N4′ for P and Q. So assume $|P| = |Q|$ and $|P + Q| = |P - Q|$.

Now suppose $a > 0$. Applying the triangle inequality to (2), we get, since P and Q are independent, the contradiction

$$|P + Q| < |P - Q|.$$

If $a < 0$, we obtain, similarly, the contradiction

$$|P + Q| > |P - Q|.$$

Hence $a = 0 = b$. Similarly, $c = d = 0$, yielding N4′. This completes the proof.

APPLICATION: Let $\sum$ be complete. Let "U is perpendicular to V," written $U \perp V$, mean $|U - tV| \geqq |U|$. Let $\sum$ be said to be metrically Euclidean if $U \perp V$ implies $|U + V|^2 = |U|^2 + |V|^2$. With these definitions it has been shown by Garrett Birkhoff,[4] if $\sum$ has not less than three dimensions, that $\sum$ is metrically Euclidean if (and, of course, only if): $U \perp V$ implies $V \perp U$, and there is at most one perpendicular from a given line to a point not on that line. These conditions are used to obtain scalar products,[5] whose presence is guaranteed by our Theorem, if N4 is satisfied. Taking advantage of their presence we see at once, by examining $f(t) = |U - tV|^2$, that $U \perp V$ if and only if $(U, V) = 0$, so that $U \perp V$ implies $V \perp U$, and also that for any U and V, $f(t)$ has only one minimum, as required. Thus N4 yields the special hypotheses of Birkhoff's Theorem. Further, our argument has taken place entirely in two dimensions, so that the requirement that $\sum$ have not less than three dimensions may be dropped.[6] We thus have

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[4] Duke Math. Jour., Vol. 1 (1935) pp. 169–172. The author is indebted to the referee for kindly bringing this paper to his attention.

[5] Given scalar products, Birkhoff refers for details to the mimeographed lecture notes of J. von Neumann on "Operator Theory," Theorem 12.27, Princeton, 1934.

[6] In this sense, N4 is stronger than Birkhoff's condition, so that our result is weaker. Whenever scalar products are present, in such a way that $|X|^2 = (X, X)$, any criterion for perpendicularity equivalent to the criterion $(U, V) = 0$ will provide the metrically Euclidean property.

COROLLARY 1: *A complete, normed, real linear space is metrically Euclidean if (and only if)* $|U| = |V|$ *implies, for any scalars a and b, that* $|aU + bV| = |bU + aV|$.

EXTENSION: Condition N4′ was derived by Jordan and von Neumann is a complex space. Our proof that N4 implies N4′ holds in a complex space. Hence N4 is sufficient for the existence of scalar products in a complex space.

To see that N4 is also necessary, let $\bar{x}$ stand for the complex conjugate of x. Due to the presence of complex scalars, we now require that $(P, Q) = \overline{(Q, P)}$, whence $(P, aQ) = \bar{a}(P, Q)$. Using these facts, we may repeat the argument preceding our theorem, whenever $|P| = |Q|$, to obtain, for any complex a and b,

$$|aP + bQ| = |\bar{b}P + \bar{a}Q|. \tag{3}$$

Now let $a = re^{i\theta}$, $b = se^{i\phi}$, with r, s, θ, and ϕ real. By (3) we may write

$$|re^{i\theta}P + se^{i\phi}Q| = |se^{-i\phi}P + re^{-i\theta}Q| = |e^{-i\theta-i\phi}||se^{i\theta}P + re^{i\phi}Q|.$$

Thus, letting $e^{i\theta}P = R$ and $e^{i\phi}Q = S$, we learn[7] that, if $|R| = |S|$,

$$|rR + sS| = |sR + rS|. \tag{4}$$

Hence N4 is necessary, completing the proof of

COROLLARY 2: *In order that a normed linear space with complex scalars may permit the definition of a scalar product, it is necessary and sufficient that, whenever* $|P| = |Q|$, *and a and b are real scalars,*

$$|aP + bQ| = |bP + aQ|.$$

As pointed out by Jordan and von Neumann, the presence of scalar products in a complex space is a necessary and sufficient condition that every subspace of dimension not greater than two be isometric to a Euclidean space. In order that a space be Euclidean, in this sense, it is thus necessary and sufficient that each two-dimensional mapping of the form $aP + bQ \to \bar{b}P + \bar{a}Q$, $|P| = |Q|$, be isometric. We have also seen that this will be the case if and only if each subspace with two real dimensions displays the axial symmetry mentioned earlier.

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[7] This argument can be reversed, of course, to derive (3) for complex and a and b from (4) for real r and s.

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THE INTRINSIC MEASURE THEORY OF RIEMANNIAN AND EUCLIDEAN METRIC SPACES

BY LYNN H. LOOMIS

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1. Introduction

Euclidean spaces are boundedly compact and have the metric property that any two spheres are similar. In particular, if a closed sphere of radius r can be covered by n open spheres of radius x, then, for any positive a, every closed sphere of radius ar can be covered by n open spheres of radius ax. We shall show that this combinatorial similarity property is sufficient in any boundedly compact metric space for the development of ordinary Lebesgue measure theory. One result is the validity of the usual formula for the volume of a sphere. That is, apart from a multiplicative constant there is one and only one measure which is a volume in the sense that spheres of equal radii have equal measures, and there is an α such that the volume of a sphere of radius r is r^α. The existence proof will be presented in a general enough form to include the development of the intrinsic measure theories of Riemannian metric spaces and of metric spaces like the Cantor sets for which the dimension α is non-integral.

2. The existence theorem

Let M be a boundedly compact metric space for which the following combinatorial congruence axiom is satisfied:

POSTULATE 1. *There is a constant K ($K \geqq 1$) such that if some closed sphere of radius r can be covered by n open spheres of radius x, then, for any positive a, every closed sphere of radius ar can be covered by n open spheres of radius Kax.*

Let $h(r, x)$ be the smallest number such that every closed sphere of radius r can be covered by $h(r, x)$ open spheres of radius x.[1] By Postulate 1, $h(r, x)$ is finite and satisfies the inequality:

$$h(ar, Kax) \leqq h(r, x). \tag{1}$$

It is also evident from the definition that

$$h(r, x) \leqq h(r, y)h(y, x). \tag{2}$$

Then $h(ar, ax) \leqq h(ar, Kax)h(Kax, ax) \leqq h(r, x)h(K^2, 1)$, so that

$$h(ar, ax) \leqq Ah(r, x), \tag{3}$$

where $A = h(K^2, 1)$.

[1] This function is thus similar to the Haar covering function. See Saks, *Theory of the Integral* (1937), p. 315.

LEMMA 1. *There is a positive constant B such that*

$$h(r, y)h(y, x) \leqq Bh(r, x) \tag{4}$$

whenever $x < y < r$.

If we define $f(r, x)$ to be the largest number such that every open sphere of radius r contains $f(r, x)$ disjoint closed spheres of radius x, then we clearly have $f(r, x) \geqq f(r, y)f(y, x)$. Some open sphere U with radius r contains $f(r, x)$ disjoint closed spheres S_i with radius x and contains no larger such set. But then (if $x < r/2$) the open spheres concentric with the S_i and with radius $3x$ cover the closed sphere concentric with U and with radius $r/2$. For if p were an uncovered point the closed sphere about p with radius x would lie in U and touch no S_i, contradicting the choice of the S_i as a maximal set. We thus infer that $f(r, x) \geqq h(r/2, 3Kx)$ whenever $x < r$, the inequality being obvious if $r/2 \leqq x < r$. Finally, $f(r, x) \leqq h(r, x)$. For if a set of open spheres of radius x covers a set of disjoint closed spheres of radius x then no open sphere can contain the center of more than one closed sphere.

These three inequalities on $f(r, x)$ imply that

$$h(r, x) \geqq f(r, x) \geqq f(r, y)f(y, x) \geqq h(r/2, 3Ky)h(y/2, 3Kx).$$

But by (1) and (2)

$$h(r, y) \leqq h(r, r/2)h(r/2, 3Ky)h(3Ky, y) \leqq [h(3K^2, 1)]^2 h(r/2, 3Ky)$$

and similarly for $h(y, x)$ so that the lemma follows with $B = [h(3K^2, 1)]^4$.

THEOREM 1. *There is a positive constant M and a unique positive[2] exponent α such that*

$$M^{-1}\left(\frac{r}{x}\right)^{\alpha} \leqq h(r, x) \leqq M\left(\frac{r}{x}\right)^{\alpha} \tag{5}$$

whenever $x \leqq r$.

The inequalities (2), (3) and (4) show that there is a positive constant C ($C > 1$) such that

$$C^{-1}h(1, a)h(1, b) < h(1, ab) < Ch(1, a)h(1, b)$$

where $a, b < 1$. If we abbreviate $h(1, a)$ as $h(a)$ and apply this inequality repeatedly to $h(a^m)$, we obtain the inequality:

$$C[C^{-1}h(a^m)]^n < h(a^{mn}) < [Ch(a^m)]^n C^{-1} \tag{6}$$

Let (6′) be the result of interchanging m and n in (6). Suppose that $m \geqq n$ so that $C^{1/m} \leqq C^{1/n}$. We then obtain from (6) and (6′), by dividing and extracting mn-th roots, that

$$C^{-2/n} < \frac{h(a^n)^{1/n}}{h(a^m)^{1/m}} < C^{2/n}$$

[2] We shall assume that the space M contains more than 1 point. Then by Postulate 1 the function $h(r, x)$ is unbounded and α must be positive.

whenever $m \geqq n$. Thus if a is fixed ($a < 1$), $h(a^n)^{1/n}$ converges to a limit l as $n \to \infty$ and $C^{-2}l^n \leqq h(a^n) \leqq C^2 l^n$. The monotonicity of $h(1, x)$ implies that for any positive y

$$(C^2 l)^{-1} l^y \leqq h(1, a^y) \leqq (C^2 l) l^y.$$

We obtain Theorem 1 from this inequality (with $M = AC^2 l$) by replacing a^y by x/r, taking α as $-\log l / \log a$ so that $l^y = (r/x)^\alpha$, and applying (3).

THEOREM 2. *If N disjoint closed spheres S_i with radii r_i lie in a sphere S with radius r, then*

$$\sum_{i=1}^{N} r_i^\alpha \leqq K^{2\alpha} r^\alpha.$$

If S is covered by $h(r, x)$ open spheres of radius x, where x is less than one-half the distance between any two S_i, then no covering sphere can touch more than one S_i. By the definition of $h(r, x)$ and Postulate 1, the number touching S_i is at least $h(r_i, Kx)$ so that $\sum^N h(r_i, Kx) \leqq h(r, x)$, or by (1),

$$\sum_{i=1}^{N} h(r, rK^2 x/r_i) \leqq \sum_{i=1}^{N} h(r_i, Kx) \leqq h(r, x). \tag{7}$$

By Theorem 1, $h(r, x)$ can be written as $M(r, x)(r/x)^\alpha$ where $M^{-1} \leqq M(r, x) \leqq M$. Then (7) implies at once that

$$\sum_{i=1}^{N} \frac{M(r, rK^2 x/r_i)}{M(r, x)} r_i^\alpha \leqq K^{2\alpha} r^\alpha.$$

If x_n ($x_n \to 0$) is chosen so that $M(r, x_n) \to \underline{\lim}_{x \to 0} M(r, x)$ as $n \to \infty$, the theorem follows.

THEOREM 3. *If the closed sphere S with radius r is covered by N open spheres S_i with radii r_i, then*

$$r^\alpha \leqq K^{2\alpha} \sum_{i=1}^{N} r_i^\alpha .$$

The proof is similar to that of Theorem 2.

We now introduce Hausdorff α-dimensional volume. The outer measure $\mu(A)$ of an arbitrary set A is defined to be the limit as $\epsilon \to 0$ of the greatest lower bound of sums $\sum^\infty r_n^\alpha$ such that 1) A is covered by a countable family S_n of open spheres with radii r_n, and 2) $r_n < \epsilon$ for all n. It follows[3] that $\mu(A)$ is a Caratheodory outer measure and the usual development of measure theory is valid. Moreover, $\mu(A)$ is a non-trivial measure, for by Theorems 1 and 3 the measure of any closed sphere S with radius r is finite and positive, satisfying in fact the inequality:

$$K^{-2\alpha} r^\alpha \leqq \mu(S) \leqq M r^\alpha. \tag{8}$$

[3] See, for example, Saks, loc. cit., pp. 53, 54.

THEOREM 4. *Postulate* 1 *is a necessary and sufficient condition that a boundedly compact metric space have a measure* $\mu(A)$ *such that for suitable positive constants* α, C_1 *and* C_2,

$$C_1 r^\alpha < \mu(S) < C_2 r^\alpha, \tag{9}$$

where S *is any sphere with radius* r.

The sufficiency was remarked above. Now suppose that $\mu(A)$ satisfies (9) and that a closed sphere of radius r is covered by n open spheres of radius x. Then $C_1 r^\alpha \leqq n C_2 x^\alpha$. Now let S be any closed sphere with radius ar. If m is the smallest number of open spheres with radius Kax required to cover S, then, as was observed in the proof of Lemma 1, the open sphere concentric with S and with twice the radius contains a set of m disjoint closed spheres with radius $Kax/3$. Hence $mC_1(Kax/3)^\alpha < C_2(2ar)^\alpha$. It follows from these two inequalities that if K is taken as $6(C_2/C_1)^{2/\alpha}$ then $m \leqq n$ and Postulate 1 holds.

It is obvious from these remarks that if Postulate 1 is satisfied and if a measure $m(A)$ has the property that there are positive constants A_1 and A_2 such that for any two closed spheres S_1 and S_2 of the same radius

$$A_1 \leqq \frac{m(S_1)}{m(S_2)} \leqq A_2,$$

then again (9) is satisfied.

To get more exact information concerning the measures of spheres we need a simple form of the Vitali covering theorem. A proof will be included for completeness though it can be read almost word for word from any standard account.[4] We suppose given a bounded set A, every point of which lies in arbitrarily small spheres of a family F of closed spheres.

THEOREM 5. *There is a sequence* S_n *of disjoint spheres of* F *such that, if* T_n *is the open sphere concentric with* S_n *and with five times the radius, then for every* N

$$A - \sum_{n=1}^{\infty} S_n \subseteq \sum_{n=N}^{\infty} T_n .$$

We restrict ourselves to the spheres of F lying in some bounded open set O which contains A. A finite number of the S_n can be chosen arbitrarily (but disjoint!) and beyond that S_m can be taken as any sphere not touching $\sum^{m-1} S_i$ whose radius is at least one-half the least upper bound δ_m of the radii of the spheres of F (in O) not touching $\sum^{m-1} S_i$.

Any point of A not in $\sum^{\infty} S_i$ is at a positive distance from $\sum^{N} S_i$, and so lies in some sphere S of F not touching $\sum^{N} S_i$. Since $\delta_i \to 0$, S must touch a first S_m $(m > N)$, so that $r(S) < \delta_m \leqq 2r(S_m)$, and S lies in T_m, concluding the proof of the theorem.

Let $m(A)$ be any measure satisfying (9). Then there is a constant B such that $m(T_n) \leqq Bm(S_n)$, and

$$m\left(A - \sum_{n=1}^{\infty} S_n\right) \leqq \sum_{n=1}^{\infty} m(T_n) \leqq B \sum_{n=N}^{\infty} m(S_n).$$

[4] See Saks, loc. cit., p. 109.

Since $\sum^{\infty} m(S_n) \leqq m(0) < \infty$, the last member approaches O as $N \to \infty$. Thus the sequence S_n covers A except for a set of measure O. If, in particular, A is open and the spheres of F lie in A, then $m(A) = \sum^{\infty} m(S_n)$.

THEOREM 6. *If S is an open sphere with radius r, then*

$$K^{-2\alpha} r^{\alpha} \leqq \mu(S) \leqq K^{2\alpha} r^{\alpha}.$$

Let S_n with radii r_n be a Vitali sequence for S chosen from the closed spheres in S. Theorem 2 applied to the partial sums implies that $\sum^{\infty} r_n^{\alpha} \leqq K^{2\alpha} r^{\alpha}$. But the sequence S_n can be enlarged to an open covering of S (by replacing S_n by T_n for $n > N$ and enlarging slightly the first N spheres) in such a way that the increase in $\sum^{\infty} r_n^{\alpha}$ is less than an arbitrary δ. Since the spheres of S_n can be taken with all radii less than any given ϵ, it follows that $\mu(S) \leqq K^{2\alpha} r^{\alpha} + \delta$, which proves the right hand inequality of Theorem 6.

Since any covering of S covers every closed sphere in S, Theorem 3 implies that $\mu(S) \geqq K^{-2\alpha} t^{\alpha}$ for every t less than r, and this completes the proof of the theorem.

3. Euclidean metric spaces

By a *Euclidean metric space* we shall mean any boundedly compact metric space in which Postulate 1 holds with $K = 1$, that is, in which the smallest number of open spheres of radius x required to cover a closed sphere of radius r depends only on the ratio x/r. Ordinary Euclidean spaces are special cases. We restate Theorem 6.

THEOREM 7. *In a Euclidean metric space of dimension α the Hausdorff α-dimensional volume of a sphere of radius r is r^{α}.*

We have thus established the ordinary formula for the volume of a sphere. We now show that, to within a multiplicative constant, this is the only possible volume.

THEOREM 8. *If $m(A)$ is a measure in M which is a volume in the sense that closed spheres of equal radius have equal measures, then for some positive k, $m(A) \equiv k\mu(A)$.*

Let O be any bounded open set. We norm the measure $m(A)$ so that $m(0) = \mu(0)$. The closed spheres in O for which $m(S) > \mu(S)$ cannot form a Vitali set for O, for otherwise we could choose a Vitali sequence S_n from them and obtain the contradiction:

$$m(0) = \mu(0) = \sum_{n=1}^{\infty} \mu(S_n) < \sum_{n=1}^{\infty} m(S_n) = m(0).$$

Hence there is a point p in O and a positive ϵ such that $m(S) \leqq \mu(S)$ for every closed sphere touching p with radius less than ϵ. But then $m(S) \leqq \mu(S)$ for every closed sphere with radius less than ϵ. Similarly $m(S) \geqq \mu(S)$ for every closed sphere in O with radius less than a suitable ϵ', so that $m(S) = \mu(S)$ for all sufficiently small closed spheres. Hence, by the Vitali theorem, $m = \mu$ for all bounded open sets and so for all measurable sets.

4. Riemannian metric spaces

The spaces of Riemannian geometry have the property that given any point p and any number d greater than 1 there is an open sphere S about p such that any two open (or closed) spheres in S are similar to within a multiplicative error of d. That is, if S_1 and S_2 with radii r_1 and r_2 are two such spheres, then there is a one-to-one mapping f such that $S_2 = f(S_1)$ and

$$d^{-1}\left(\frac{r_1}{r_2}\right) \leqq \frac{\rho(p, q)}{\rho(f(p), f(q))} \leqq d\left(\frac{r_1}{r_2}\right)$$

for every pair of points p and q in S_1. Thus ordinary Riemannian spaces are included in the following definition. A *Riemannian metric space* is a connected locally compact metric space in which Postulate 1 holds with any value of K greater than 1 for the closed spheres contained in a sufficiently small fixed sphere about any given point.[5] A Riemannian metric space is *locally Euclidean* if Postulate 1 holds locally with $K = 1$.

The existence theory of section 2 can now be applied only locally. Let U be a sphere (radius r_0) in which Postulate 1 holds and let V be the concentric closed sphere with radius $r_0/3$. We define $h(r, x)$ with $x \leqq r \leqq r_0/3$ in terms of coverings of $S \frown V$ where S is the general closed sphere with radius r, and we define $f(r, x)$ in terms of packings of spheres contained in V. Then (1) and (2) hold and the rest of section 2 applies automatically if only spheres lying interior to V are considered. We assume the space to be connected in order to insure that the resulting dimension α is the same over the whole space. This effect can also be obtained, without postulating connectedness, by assuming that Postulate 1 holds for any K greater than 1 in any compact set, provided that only spheres are considered having radii less than a suitable ϵ. With this common value for α the Hausdorff α-dimensional volume is defined for all separable Borel sets and is finite and non-zero for every open set with compact closure. The following theorem is evident from the definition of Riemannian metric spaces and Theorem 6.

THEOREM 9. *If F is a compact set in a Riemannian metric space of dimension α, then the Hausdorff α-dimensional volumes of spheres in F with radius r are uniformly asymptotic to r^α as $r \to 0$.*

Similarly, the formula for the volume of a sphere holds in the small in a locally Euclidean space.

THEOREM 10. *If $m(A)$ is a measure for which closed spheres of equal radius in a compact set have asymptotically equal measures as $r \to 0$, then there is a positive constant k such that $m(A) \equiv k\mu(A)$.*

The proof of Theorem 8 is general and shows that there is some point p in O such that $m(S) \leqq \mu(S)$ for all sufficiently small closed spheres touching p and a second point q for which the inequality is reversed. It follows that $m(S)$ is

[5] Compare this definition with the properties considered by Busemann, *Metric Methods in Finsler Spaces and in the Foundations of Geometry*, p. 40, (1), (1a) and (1b).

asymptotically equal to $\mu(S)$ as $r(S) \to 0$, and by the Vitali theorem that $m = \mu$ in O.

Thus section 2 contains a development of the intrinsic measure theory of Riemannian metric spaces, an application being a coordinate-free development of measure theory in ordinary Riemannian spaces.

5. Cantor spaces

In this section we shall assume Postulate 1 in the Euclidean form for a restricted set of spheres. Let M be a boundedly compact metric space and let O be a family of open spheres in M with radii of the form ba^n for fixed a and b $(0 < a < 1)$. We suppose that i) there is a positive constant C such that any sphere in M with radius Ca^n $(n \geqq 1)$ lies in a sphere of O with radius ba^n, and that ii) the smallest number of spheres of O with radius x required to cover the closure of a sphere of O with radius r depends only on the ratio x/r. The theory of section 2 can be applied to the spheres of O. In the proof of Lemma 1 the concentric spheres of radius $3x$ may not be in O, but by property i) above, each such sphere lies in a sphere of O with radius at most $(b/Ca)3x$. The outer measure μ is now defined using coverings by spheres from O. Thus if S is the closure of a sphere in O with radius ba^n, then (inequality (8))

$$(ba^n)^\alpha \leqq \mu(S) \leqq M(ba^n)^\alpha.$$

Now let S be any sphere in M whose radius x is at most Ca. For some n, $2ba^n < x \leqq 2ba^{n-1}$ so that if S_1 is the closure of any sphere of O which contains the center of S and has radius ba^n, then S_1 lies in S. Also there is a sphere S_2 of O which contains S and has radius at most bx/Ca. Hence

$$\left(\frac{a}{2}\right)^\alpha x^\alpha \leqq (ba^n)^\alpha \leqq \mu(S_1) \leqq \mu(S) \leqq \mu(S_2) \leqq M\left(\frac{b}{Ca}\right)^\alpha x^\alpha.$$

Thus (9) holds, and μ is a measure for which the Vitali theorem is valid.

The ordinary Cantor sets are spaces of this type. Let M_0 be the closed unit interval $[0, 1]$. We suppose inductively that M_n consists of 2^n closed intervals of length $((1 - l)/2)^n$ and we form M_{n+1} from M_n by removing the open interval of length $l((1 - l)/2)^n$ from the center of each closed interval of M_n. Then $M = \Pi^\infty M_i$ is the ordinary Cantor set. The 2^{n+1} endpoints of the 2^n open intervals removed from M_n make up those endpoints of M_{n+1} which are not endpoints of M_n. Let O_{n+1} be the set of 2^{n+1} open spheres in M with these points as centers and with radius $((1 - l)/2)^{n+1} + kl((1 - l)/2)^n$ $(= D((1 - l)/2)^n)$ where $1 < k < ((1 - l)/2)^{-1}$. Each interval of M_n is covered by either of the two spheres of O_{n+1} having endpoints in it, and M (and M_n) can be covered by 2^n of these spheres but by no fewer. Any sphere of radius less than $kl((1 - l)/2)^n$ lies in a sphere of O_{n+1}. Finally any two spheres of $O = \sum^\infty O_n$ are similar. Thus properties i) and ii) are satisfied, and it remains only to compute the dimension α of the space.

By Theorem 1, $M^{-1}x^{-\alpha} \leqq h(1, x) \leqq Mx^{-\alpha}$ and we have seen that $h(1, x) = 2^n$ if $x = D((1 - l)/2)^n$. Hence

$$M^{-1}D^{-\alpha}\left(\frac{1-l}{2}\right)^{-n\alpha} \leqq 2^n \leqq MD^{-\alpha}\left(\frac{1-l}{2}\right)^{-n\alpha}.$$

Taking nth roots and letting $n \to \infty$ we find that $((1 - l)/2)^{-\alpha} = 2$, i.e., that $\alpha = \log 2 / \log (2/(1 - l))$. If M is formed by removing middle thirds ($l = \frac{1}{3}$) then $\alpha = \log 2 / \log 3$.

There seems to be no way in which the general measure of this section can be considered a unique volume. Let us now replace ii) by the following stronger property: iii) if the closure of some sphere of O with radius r_1 can be covered by N spheres of O with radii x_i, $i = 1, \cdots, N$, then the closure of any sphere of O with radius r_2 can be covered by N spheres of O with radii $(r_2/r_1)x_i$. It follows from iii) that if S_1 and S_2 are the closures of spheres in O with radii ba^n and ba^m then $\mu(S_2) = a^{(m-n)\alpha}\mu(S_1)$. Therefore if μ is multiplied by a suitable positive constant, $\mu(S) = r^\alpha$ where S is the closure of any sphere in O and r is its radius. Thus μ can be considered a volume with respect to these spheres. Finally, since the closures of spheres in O form a Vitali set for the space, Theorem 8 can be applied and shows that μ is essentially (to within a multiplicative constant) the only such volume.

The replacement of ii) by iii) can certainly be made whenever, as in the case of the Cantor sets or Euclidean spaces, the spheres of O are all similar.

6. Cartesian products

Let M_1 and M_2 with dimensions α and β be boundedly compact metric spaces satisfying Postulate 1 and let M be their Cartesian product with distance defined by the usual Euclidean formula. Then M satisfies Postulate 1 and its dimension is $\alpha + \beta$. This follows readily from the fact that any sphere in M with radius r contains the "square" whose sides are spheres in M_1 and M_2 with radius $r/2^{\frac{1}{2}}$, and is contained in the square whose sides have radius r. Hence

$$h(r, \sqrt{2}x) \leqq h_1(r, x)h_2(r, x)$$

$$f(r, x) \geqq f_1(r/\sqrt{2}, x)f_2(r/\sqrt{2}, x),$$

where f is the packing function of Lemma 1, and the result is immediate.

If M_1 and M_2 are spaces in which closed spheres are similar then M has the same property. Thus $K_1 = K_2 = K = 1$ and all three spaces are Euclidean metric spaces.

But in general it does not follow, at least from the above argument, that $K = 1$ whenever $K_1 = K_2 = 1$. Nevertheless it is easily verified by integration that the volume of a sphere in M with radius r is proportional to $r^{\alpha+\beta}$. Thus is raised the problem of determining combinatorial conditions in a boundedly compact metric space which are necessary and sufficient for the existence of a measure which has the value r^α on a sphere of radius r.

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UNIFORM CONVEXITY IN FACTOR AND CONJUGATE SPACES

BY MAHLON M. DAY

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1. Introduction.

In a recent series of short papers [2, 3, 4] I have been discussing the relationships of uniform convexity with certain other properties of normed vector spaces. In this paper I propose to discuss relationships between uniform convexity, factor spaces, and conjugate spaces. Because of the large number of special results which are needed for two dimensional spaces—and which are false in general—the paper will be divided into two parts: In Part I (§§2–4) B is two dimensional; in Part II (§§5–7) this restriction is removed and B may be any normed vector space.

To recall the definition, B is said to be *uniformly convex* if there exists a function δ such that $0 < \delta(\epsilon)$ if $0 < \epsilon \leqq 2$ and such that $\|b_1 + b_2\| \leqq 2(1 - \delta(\epsilon))$ if $\|b_1\| = \|b_2\| = 2$ and $\|b_1 - b_2\| \geqq \epsilon$; such a function δ is called a *modulus of convexity* for B. It is easily seen that B is uniformly convex if and only if the special monotone function δ defined by $2\delta(\epsilon) = \inf [2 - \|b_1 + b_2\|]$, where the infimum is taken for $\|b_1\| = \|b_2\| = 1$ and $\|b_1 - b_2\| \geqq \epsilon$, is a modulus of convexity for B. Note that if δ_1 is nowhere greater than δ, a modulus of convexity for B, then δ_1 is a modulus of convexity for B.

Because of the pointwise nature of the definition of uniform convexity, it is clear that B is uniformly convex with modulus of convexity δ if and only if all the two dimensional subspaces of B have the common modulus of convexity δ. One of the most useful results of this investigation (Theorem 5.5) is the complementary fact that B is uniformly convex if and only if all the two dimensional factor spaces of B have a common modulus of convexity.

In the study of the effect of uniform convexity of B on the nature of the conjugate space B^*, this result makes it possible to reduce the problem to the study of two dimensional spaces. In such a space B uniform convexity is equivalent to strict convexity: that is, a two (or finite) dimensional space is uniformly convex if and only if there does not exist a line segment of positive length all of whose points are of norm one. It has been observed [1, Footnote 13] that such a line segment on the unit sphere is equivalent to the existence of a "sharp edge" on the unit sphere in B^*. The attempt to describe a "sharp edge" of the unit sphere in a finite dimensional space B in terms of the norm in that space leads to the condition that there exists a $k > 0$ such that for any $\epsilon > 0$ a pair of points b_1 and b_2 exists such that $\|b_1 - b_2\| < \epsilon$ while $2 - \|b_1 + b_2\| > k\|b_1 - b_2\|$. Contradicting this suggests the following condition, a sort of dual concept to uniform convexity. A space B is said to be *uniformly flattened* if there exists a function η positive for $0 < \epsilon \leqq 2$ such that $\lim_{\epsilon \to 0} \eta(\epsilon) = 0$ while $(2 - \|b_1 + b_2\|)/\|b_1 - b_2\| \leqq \eta(\epsilon)$ if $\|b_1\| = \|b_2\| = 1$ and $\|b_1 - b_2\| \leqq \epsilon$;

such a function is called a *modulus of flattening* for B. Again it is clear that B is uniformly flattened if and only if the special monotone function η defined by $2\eta(\epsilon) = \sup (2 - \| b_1 + b_2 \|)/\| b_1 - b_2 \|$, where the supremum is taken over all b_i for which $\| b_1 \| = \| b_2 \| = 1$ and $\| b_1 - b_2 \| \leqq \epsilon$, is a modulus of flattening for B. Also note that if η_1 is nowhere less than η, a modulus of flattening for B, then η_1 is a modulus of flattening for B.

The corresponding functions defined in the same way in B^* will be denoted by δ^* and η^*. Note that the geometrical interpretation of the modulus of convexity δ in B is that every chord of the unit sphere of length at least ϵ has its midpoint at a depth at least $\delta(\epsilon)$ below the surface of the unit sphere. The existence of a modulus of flattening η for B means that the ratio of depth of midpoint to length of a chord of the unit sphere approaches 0 with the chord length.

The principal result (Theorem 6.7) toward which this whole paper is directed is the theorem that B is uniformly convex (flattened) if and only if B^* is uniformly flattened (convex). §7 is an application of 5.5 to the problem considered in [2] and [4].

Part I: Two Dimensional Spaces

2. Two geometrical lemmas.

To save later circumlocutions it will be convenient to define the phrase "b is between b_1 and b_2" for points of norm one to mean that b is on that closed arc of $\{b' \mid \| b' \| = 1\}$ which has end points b_1 and b_2 and which does not contain $-b_1$ and $-b_2$. (This arc is unique only if $b_1 \neq -b_2$; the case $b_1 = -b_2$ will cause no trouble in the computation to follow.) Note that if $b_1 \neq -b_2$, then $(b_1 + b_2)/\| b_1 + b_2 \|$ is between b_1 and b_2.

2.1 Lemma. *If b is between b_1 and b_2 and if $b_3 = (b_1 + b_2)/2$, then $\| b - b_3 \| \leqq \| b_1 - b_2 \|/2$; if $\| b_1 - b_2 \| < 2$ and b is not between b_1 and b_2, then $\| b - b_3 \| > \| b_1 - b_2 \|/2$.*

This is obvious if $b_1 = -b_2$; if not, $b_4 = b_3/\| b_3 \|$ is defined and lies between b_1 and b_2. Hence b lies between b_1 and b_4 or between b_4 and b_2; by an interchange of subscripts if necessary it can be assumed that the first case holds. If $b_5 = (b_1 - b_3)/\| b_1 - b_3 \|$ and $b_6 = (b - b_3)/\| b - b_3 \|$, then b_6 lies between b_5 and b. Set a coordinate system on the space so that $b_5 = (-1, 0)$ and $b_4 = (0, 1)$; then every chord between points on the arc from b_5 to b has an inclination between the inclination α of any line of support at b_5 and $\alpha - \pi$. Letting incl $b'b''$ be the inclination of such a chord, we have by convexity of the unit sphere that incl $b_5b_6 \geqq$ incl $b_5b \geqq$ incl b_1b. However, for b between b_1 and b_4, the assertion that $\| b - b_3 \| > \| b_1 - b_3 \|$ is equivalent to incl $b_1b >$ incl b_5b_6. This shows that $\| b - b_3 \| \leqq \| b_1 - b_3 \| = \| b_1 - b_2 \|/2$.

A proof of the other part of the lemma is of the same nature so it need not be given here, especially as we shall not apply it.

2.2 Corollary. *If b and b' are between b_1 and b_2, then $\| b - b' \| \leqq \| b_1 - b_2 \|$.*

2.3 COROLLARY. *If δ and η are the special functions of §1, then $\delta(\epsilon) \leqq \epsilon/2$ and $\eta(\epsilon) \leqq 1/2$ for all ϵ.*

If $\| b_1 - b_2 \| = \epsilon$, then $\delta(\epsilon) \leqq 1 - \| b_3 \| = \| b_4 - b_3 \| \leqq \epsilon/2$ and $\eta(\epsilon) = \sup_{\| b_1 - b_2 \| \leqq \epsilon} \| b_4 - b_3 \| / \| b_1 - b_2 \| \leqq \| b_1 - b_2 \| / 2 \| b_1 - b_2 \| = 1/2$.

2.4 LEMMA. *If b is given on the unit sphere and $0 < \epsilon < 2$, there is precisely one pair of points b_1, b_2 on the unit sphere such that $b = (b_1 + b_2)/\| b_1 + b_2 \|$ while $\| b_1 - b_2 \| = \epsilon$.*

Let β be an element of B^* of norm one such that $\beta(b) = 0$, and let b_0 be that point on the unit sphere of B such that $\beta(b_0) = 1$ while $\beta(b') < 1$ if $b' \neq b_0$ is between b_0 and b. For any such b' excluding the point $-b'$ there is just one point b'' satisfying the equations $\| b'' \| = 1$ and $\beta(b'') = -\beta(b')$; $\| b' - b'' \|$ is a continuous function of $\beta(b')$ which is non-decreasing by 2.2 and which approaches 2 as $\beta(b')$ approaches 1 since $\| b' - b'' \| \geqq \beta(b' - b'') = 2\beta(b')$. Moreover $\| b' - b'' \|$ is strictly increasing with $\beta(b')$; if it were not then there would exist two non-intersecting chords, $b_1'b_1''$ and $b_2'b_2''$, of the same length k with both b_1' and b_2' between b and b_0. Then by 2.2 every chord $b'b''$ with b' between b_1' and b_2' and b'' between b_1'' and b_2'' has the same length k, so we can assume that the original two chords are parallel. Then $\beta(b_1') = \beta(b_2'')$ so must equal 1 by convexity. It follows that there exists a pair b', b'' such that $\| b' - b'' \| = \epsilon$; clearly $(b' + b'')/\| b' + b'' \| = b$ or $-b$; the last case is impossible by an argument similar to the one above so this pair of points satisfies the conditions of the lemma.

3. An estimate of δ in terms of η^*.

Some of the formal computations of this section were originally suggested by a geometrical picture which may be kept in mind to help clarify the arguments. If the distances are prescribed between pairs of a set of three non-collinear points in a two dimensional normed vector space, then six points on the unit sphere are determined and the set of six lines through adjacent pairs of these points describes a six-pointed star which contains the unit sphere. Some estimates of the norm of a β in B^* will be obtained by calculating the upper bound over part of this star of $\beta(b)$.

3.1 LEMMA. *If $\| b_1 - b_2 \| < 1$, if $\beta(b_1) \cdot \beta(b_2) \leqq 0$, if $\| \beta \| \neq 0$ and if b is a point of norm one for which $\beta(b) = \| \beta \|$, then neither b nor $-b$ is between b_1 and b_2.*

Since $\beta(b_1) \cdot \beta(b_2) \leqq 0$, there is a point b' between b_1 and b_2 such that $\beta(b') = 0$. If b or $-b$ lies between b_1 and b_2, then $\| \beta \| = | \beta(b' \pm b) | \leqq \| \beta \| \cdot \| b' \pm b \| < \| \beta \|$ by 2.2.

3.2 LEMMA. *If $\| b_i \| = \| \beta_i \| = \beta_i(b_i) = 1$, $i = 1, 2$, and if $\| b_1 - b_2 \| < 1$, then*

$$\| \beta_1 - \beta_2 \| \cdot \| b_1 - b_2 \| \leqq 2(\beta_1 - \beta_2)(b_1 - b_2) = 2[2 - \beta_1(b_2) - \beta_2(b_1)].$$

To prove the first inequality we show that if β is any element of B^* for which $\beta(b_1) \cdot \beta(b_2) \leqq 0$, then $\| \beta \| \cdot \| b_1 - b_2 \| \leqq 2 | \beta(b_1 - b_2) |$. This is obvious if

$\| \beta \| = 0$ or if $b_1 = b_2$; if not, let $b_5 = (b_2 - b_1)/\| b_2 - b_1 \|$. Let β', β'', β''' be those elements of B^* for which $\beta'(b_1) = \beta'(b_2) = 1$, $\beta''(b_2) = \beta''(b_3) = 1$, and $\beta'''(b_3) = \beta'''(-b_1) = 1$; let b_4 be the point for which $\beta'(b_4) = \beta'''(b_4) = 1$ and b_4' the point where $\beta''(b_4') = -\beta'(b_4') = 1$. (See the diagram suggested at the head of this section.) Then by 3.1 $\sup [| \beta(b_4) |, | \beta(b_4') |] \geqq | \beta(b) | = \| \beta \|$; by a judicious arrangement of the subscripts 1 and 2 and the sign of β it may be assumed that $\beta(b_4) \geqq \| \beta \|$.

Now write $b_4 = \lambda b_1 + \mu b_3$; then $\beta'(b_3) = 0$ so $\lambda = \beta'(b_4) = 1$. Also $1 = \beta'''(b_4) = \beta'''(b_1) + \mu\beta'''(b_3) = -1 + \mu$, so $\mu = 2$ and $b_4 = b_1 + 2b_3$. Since $\beta(b_1) \cdot \beta(b_2) \leqq 0$, there is a point b_5 on the line segment from b_1 to b_2 such that $\beta(b_5) = 0$; if $b_4 = \lambda' b_5 + \mu' b_3$, again $\lambda' = 1$ while $\mu' \leqq \mu = 2$; hence $\| \beta \| \leqq \beta(b_4) = 0 + \mu'\beta(b_3) \leqq 2\beta(b_5) = 2\beta(b_2 - b_1)/\| b_2 - b_1 \|$. Therefore $\| \beta \| \cdot \| b_2 - b_1 \| \leqq 2\beta(b_2 - b_1)$.

Since $\| \beta_i \| = \| b_i \| = 1$, $(\beta_1 - \beta_2)(b_1) = 1 - \beta_2(b_1) \geqq 0$ while $(\beta_1 - \beta_2)(b_2) = \beta_1(b_2) - 1 \leqq 0$, so $\beta_1 - \beta_2$ satisfies the hypotheses for β.

3.3 LEMMA. *If* $\| b_i \| = 1$, $i = 1, \cdots, 4$, *if* $\| b_3 - b_4 \| = \epsilon > 0$ *and* $\| b_3 + b_4 \|/2 = 1 - \mu$ *where* $\mu \leqq \epsilon/8$, *if* $\| b_i - [(b_3 + b_4)/\| b_3 + b_4 \|] \| \leqq \epsilon/8$ *and* $\| \beta_i \| = \beta_i(b_i) = 1$ *for* $i = 1, 2$, *then* $1 - \beta_i(b_j) \leqq 16\mu \| b_1 - b_2 \|/\epsilon$ *for* $i, j = 1, 2$.

By a norm-preserving linear transformation (which affects neither hypotheses nor conclusions of the lemma) a rectangular coordinate system can be set up in which $b_5 = (b_4 - b_3)/\| b_4 - b_3 \|$ is the point (1, 0) and $b_6 = (b_5 + b_4)/\| b_3 + b_4 \|$ is the point (0, 1). Label the points in the order b_3 , b_1 , b_2 , b_4 along the arc from b_3 to b_4 and let C be the set of points at distance $\epsilon/8$ from b_6 which lie below the line through b_6 and b_4 ; let b_7 be that end point of C in the second quadrant. Let L be a line of support of the unit sphere at b_3 ; then incl $b_3b_1 \leqq$ incl L. If b_7 lies on the opposite side of L from b_1 , then incl $b_3b_7 >$ incl $L \geqq$ incl b_3b_1. (As in 2.1 the inclination is between some angle $\alpha < \pi$ and $\alpha - \pi$.) If b_7 lies on the same side of L as b_1 , either the line segment b_3b_7 cuts C in a point $b_8 \neq b_7$ or it does not. If not, b_3b_1 cuts C in just one point and incl $b_3b_1 <$ incl b_3b_7. The other case can not occur; if b_3b_7 cuts C in a point $b_8 \neq b_7$, there is a chord of the unit sphere through the points $(b_7 - b_6)/\| b_7 - b_6 \|$ and $(b_8 - b_6)/\| b_8 - b_6 \|$ which is of smaller inclination than the line L. Since $(b_7 - b_6)/\| b_7 - b_6 \|$ is between $-b_5$ and b_6 , this forces $(b_7 - b_6)/\| b_7 - b_6 \|$ to lie between b_3 and b_6 so the slope of $b_7b_6 \leqq$ slope of $-b_3 = -2(1 - \mu)/\epsilon$; however the slope of $b_7b_6 =$ slope of b_4b_6 is precisely $-2\mu/\epsilon$ so μ must be at least $1/2$ if b_8 exists, but $\mu \leqq \epsilon/8 \leqq 2/8$.

This proves that incl $b_5b_1 \leqq$ incl b_3b_7 ; we now show that b_5b_7 is of slope $\leqq 4\mu/\epsilon$. If (x, y) is the coordinate pair for b_7 , then b_5 is $(-\epsilon/2, 1 - \mu)$ and the slope of b_3b_7 is $(y - 1 + \mu)/(x + \epsilon/2)$. Dropping the perpendicular from b_7 to the segment b_3b_4 and using similar triangles and the fact that $k = \| b_4 - b_6 \| \geqq \epsilon/2 - \mu \geqq 3\epsilon/8$, we have $y - 1 + \mu \leqq \mu(\epsilon/8 + k)/k \leqq 4\mu/3$. Also $-x + \epsilon/2 \leqq \epsilon(\epsilon/8 + k)/2k \leqq 2\epsilon/3$ so $\epsilon/2 + x \geqq \epsilon/3$ and the slope of $b_3b_7 \leqq 4\mu/\epsilon$. Hence the slope of $b_3b_1 \leqq 4\mu/\epsilon$ and slope $b_1b_2 \leqq$ slope $b_3b_2 \leqq$ slope $b_3b_1 \leqq 4\mu/\epsilon$ also.

The same argument carried through at the other end of the interval b_3b_4 shows that slope $b_1b_2 \geqq -4\mu/\epsilon$.

Since $L' = \{b \mid \beta_1(b) = 1\}$ is a line of support of the unit sphere at b_1, slope $L' \leqq$ slope $b_3b_1 \leqq 4\mu/\epsilon$. Let b be that point of L' directly above b_2 and let b' be the point on the same vertical line as b_2 and on the same horizontal line as b_1; then slope b_1b = slope $L' \leqq 4\mu/\epsilon$ so $\| b - b' \| \leqq 4\mu \| b_1 - b' \|/\epsilon$. Also $\| b_2 - b' \| \leqq 4\mu \| b_1 - b' \|/\epsilon$ so $\| b - b_2 \| \leqq 8\mu \| b_1 - b' \|/\epsilon$. However $\| b_1 - b' \| (1 - 4\mu/\epsilon) \leqq \| b_1 - b_2 \|$, so $\| b - b_2 \| \leqq 8\mu \| b_1 - b_2 \|/\epsilon(1 - 4\mu/\epsilon) \leqq 16\mu \| b_1 - b_2 \|/\epsilon$ since $\mu \leqq \epsilon/8$.

Since $\| \beta_1 \| = 1$, $1 - \beta_1(b_2) = \beta_1(b_1 - b_2) = \beta_1(b - b_2) \leqq \| b - b_2 \| \leqq 16\mu \| b_1 - b_2 \|/\epsilon$. By interchange of subscripts the same is true for $1 - \beta_2(b_1)$.

3.4 COROLLARY. *Under the hypotheses of* 3.3 *if* $b_1 \neq b_2$, *then* $\| \beta_1 - \beta_2 \| \leqq 64\mu/\epsilon$.

Since $\| b_3 - b_4 \| \leqq 2$, $\| b_1 - b_2 \| \leqq \epsilon/4 \leqq 1/2 < 1$, so the hypotheses of 3.2 are also satisfied; hence
$\| \beta_1 - \beta_2 \| \cdot \| b_1 - b_2 \| \leqq 2(1 - \beta_1(b_2) + 1 - \beta_2(b_1)) \leqq 4[16\mu \| b_1 - b_2 \|/\epsilon]$. Hence, if $b_1 \neq b_2$, $\| \beta_1 - \beta_2 \| \leqq 64\mu/\epsilon$.

Note that even in case $b_1 = b_2$ the same result can be proved by an approximation argument; for brevity, it is not given here since this case can be avoided in the application of the lemma.

As is usual, if f is a monotone function defined on the interval $0 < x \leqq 2$, let $f^{-1}(t)$ be the supremum of those x in the interval for which $f(x) \leqq t$.

3.5 LEMMA. *If* B^* *has the modulus of flattening* η^*, *if* $\| b_3 \| = \| b_4 \| = 1$, *if* $\| b_3 - b_4 \| = \epsilon > 0$ *and* $\| b_3 + b_4 \|/2 = 1 - \mu$, *then* $\mu > \epsilon\eta^{*-1}(\epsilon/1024)/1024 = \delta_1(\epsilon)$.

If $\mu > \epsilon/8$, $\mu > \delta_1(\epsilon)$; if $\mu \leqq \epsilon/8$, choose b_1 and b_2 as far apart as the restrictions of 3.3 allow; then $\| b_1 - b_2 \| \geqq \epsilon/8$ by 2.2, while by 3.4 $\| \beta_1 - \beta_2 \| \leqq 64\mu/\epsilon$. It can easily be verified that uniform flattening of B^* implies strict convexity of B; this in turn implies that if $\beta_i(b_i) = \| \beta_i \| = \| b_1 \| = 1$, then $\beta_1 \neq \beta_2$ when $b_1 \neq b_2$. Hence, by 3.4, $0 < \epsilon^* = 8 \| \beta_1 - \beta_2 \| \leqq 512\mu/\epsilon$. Now $\delta_1(\epsilon) \leqq \epsilon/512$; hence if $\mu > \epsilon/512$, $\mu > \delta_1(\epsilon)$. If $\mu \leqq \epsilon/512$, then $\epsilon^* \leqq 1$, and, by 2.4 applied to B^*, there exist β_3 and β_4 of norm one such that $\| \beta_3 - \beta_4 \| = \epsilon^*$ and $(\beta_3 + \beta_4)/\| \beta_3 + \beta_4 \| = (\beta_1 + \beta_2)/\| \beta_1 + \beta_2 \|$; by 2.2, β_1 and β_2 are within $\epsilon^*/8$ of $(\beta_3 + \beta_4)/\| \beta_3 + \beta_4 \|$.

If $\mu^* = 1 - \| \beta_3 + \beta_4 \|/2$, then $\mu^*/\epsilon^* \leqq \eta^*(\epsilon^*) \leqq \eta^*(512\mu/\epsilon)$. Now if $\mu > \epsilon\eta^{*-1}(1/8)/1024$, $\mu > \delta_1(\epsilon)$ since $\epsilon/512 < 1/8$. If $\mu \leqq \epsilon\eta^{*-1}(1/8)/1024$, then $512\mu/\epsilon < 1024\mu/\epsilon \leqq \eta^{*-1}(1/8)$ or $\mu^*/\epsilon^* \leqq \eta^*(512\mu/\epsilon) \leqq 1/8$. Hence 3.4 applied with B^* and B interchanged gives $\epsilon/8 \leqq \| b_1 - b_2 \| \leqq 64\mu^*/\epsilon^* \leqq 64\eta^*(512\mu/\epsilon) < 128\eta^*(512\mu/\epsilon)$; that is, $512\mu/\epsilon \geqq \eta^{*-1}(\epsilon/1024)$ or $\mu \geqq \epsilon\eta^{*-1}(\epsilon/1024)/512 > \delta_1(\epsilon)$.

3.6 THEOREM. *If a two dimensional space* B^* *is uniformly flattened,* B *is uniformly convex and* δ_1, *a modulus of convexity in* B, *is determined by* η^*, *the modulus of flattening in* B^*, *from the equation* $\delta_1(\epsilon) = \epsilon\eta^{*-1}(\epsilon/1024)/1024$.

Since a two dimensional space is reflexive, note that B and B^* can be inter-

changed here. The proof of this theorem when contrasted with that of the complementary Theorem 4.3 seems to be surprisingly complicated; however I have not been able to discover a simpler one.

4. An estimate of δ in terms of η^*.

This estimate is considerably simpler to derive.

4.1 LEMMA. *If* $\| b_i \| = \| \beta_i \| = \beta_i(b_i) = 1$ *for* $i = 1, 2$, *then*

$$\| \beta_1 + \beta_2 \| \cdot \| b_1 + b_2 \| + \| \beta_1 - \beta_2 \| \cdot \| b_1 - b_2 \| \geqq 4.$$

Clearly

$$(\beta_1 + \beta_2)(b_1 + b_2) = 2 + \beta_1(b_2) + \beta_2(b_1)$$

and

$$(\beta_1 - \beta_2)(b_1 - b_2) = 2 - \beta_1(b_2) - \beta_2(b_1)$$

so

$$(\beta_1 + \beta_2)(b_1 + b_2) + (\beta_1 - \beta_2)(b_1 - b_2) = 4.$$

Hence

$$\| \beta_1 + \beta_2 \| \cdot \| b_1 + b_2 \| + \| \beta_1 - \beta_2 \| \cdot \| b_1 - b_2 \| \geqq 4.$$

4.2 LEMMA. *If* B^* *has modulus of convexity* δ^*, *if* $\| b_1 \| = \| b_2 \| = 1$ *and if* $\| b_1 - b_2 \| = \epsilon > 0$, *then* $(2 - \| b_1 + b_2 \|)/\| b_1 - b_2 \| \leqq \delta^{*-1}(\epsilon)$ *and* $\delta^{*-1}/2$ *is a modulus of flattening for* B.

The inequality holds if $b_1 = -b_2$, since $\delta^{*-1}(2) = 2$; if $b_1 \neq -b_2$, take β_i to satisfy 4.1 and let $\beta = \beta_1 - \beta_2$, $b = b_1 - b_2$; then

$$\begin{aligned} \| \beta_1 + \beta_2 \| &\geqq (4 - \| \beta \| \cdot \| b \|)/\| b_1 + b_2 \| \\ &= 2[(2 - \| \beta \| \cdot \| b \|/2)/\| b_1 + b_2 \|] \\ &= 2[1 - (\| b_1 + b_2 \|/2 - 2 + \| \beta \| \cdot \| b \|/2)/\| b_1 + b_2 \|] \\ &= 2(1 - \alpha). \end{aligned} \tag{1}$$

Consider now the inequality $(x - 2 + ky)/x \leqq y$, where $0 < x \leqq 2, 0 < y \leqq 2$ and $0 \leqq k \leqq 1$. This is equivalent to $x(1 - y) \leqq 2 - ky$ which is obviously true for $y = 1$; for $0 < y < 1$ it is true if $x \leqq (2 - y)/(1 - y)$ is true, but $(2 - y)/(1 - y) > 2$ if $0 < y < 1$. For $1 < y \leqq 2$ it is true if $x \geqq (2 - y)/(1 - y)$ is true, but $(2 - y)/(1 - y) \geqq 0$ so this also is true. Substituting $\| b_1 + b_2 \|$, ϵ and $\| \beta \|/2$ for x, y and k, this shows that $\alpha < \epsilon$.

By (1) and uniform convexity of B^*, $\| \beta \| \leqq \delta^{*-1}(\alpha) \leqq \delta^{*-1}(\epsilon)$. Applying 4.1 again

$$\| b_1 + b_2 \| \geqq (4 - \| \beta \| \cdot \| b \|)/\| \beta_1 + \beta_2 \| \geqq [4 - \epsilon\delta^{*-1}(\epsilon)]/\| \beta_1 + \beta_2 \|$$

so

$$2 - \| b_1 + b_2 \| \leqq [2 \| \beta_1 + \beta_2 \| - 4 + \epsilon\delta^{*-1}(\epsilon)]/\| \beta_1 + \beta_2 \|.$$

Factor out a two and make the substitutions x for $\| \beta_1 + \beta_2 \|$, y for $\epsilon\delta^{*-1}(\epsilon)/2 = \epsilon$, and k for 1; the inequality above shows that this right hand side $\leqq \epsilon\delta^{*-1}(\epsilon)$ or

$$(2 - \| b_1 + b_2 \|)/\| b_1 - b_2 \| < \delta^{*-1}(\epsilon).$$

Since δ^* is monotone and $0 < \delta^*(\epsilon)$ if $0 < \epsilon \leqq 2$, $\lim_{\epsilon \to 0} \delta^{*-1}(\epsilon) = 0$, so $\delta^{*-1}/2$ is a modulus of flattening for B.

Note that by 2.3 η_1 defined by $\eta_1(\epsilon) = \inf [\delta^{*-1}(\epsilon)/2, 1/2]$ is also a modulus of flattening for B.

4.3 THEOREM. *If a two dimensional space B^* is uniformly convex, then B is uniformly flattened and η_1, a modulus of flattening for B, is determined by δ^*, the modulus of convexity for B^*, from the equation* $\eta_1(\epsilon) = \inf [\delta^{*-1}(\epsilon)/2, 1/2]$.

As in 3.6 it is obvious that B and B^* may be interchanged.

PART II: GENERAL NORMED VECTOR SPACES

5. Factor spaces and uniform convexity.

This section contains the theorem on factor spaces referred to in the introduction. First, recall the definition: If B is a normed vector space and B_0 is a closed linear subspace of B, the factor space $A = B/B_0$ is the space of all cosets $a_b = \{b' \mid b' - b \in B_0\}$. The vector operations are defined as usual so that $a_b + a_{b'} = a_{b+b'}$ and $\lambda a_b = a_{\lambda b}$; for any a in A, let $\| a \| = \inf_{b \in a} \| b \|$. Then A is a normed vector space under these operations and it is easily shown that A is complete if B is. If B_0 is a subset of B, define $B_0^\perp$ to be $\{\beta \mid \beta(b) = 0$ for every b in $B_0\}$; similarly for B' in B^*, let $B'_\perp = \{b \mid \beta(b) = 0$ for every β in $B'\}$. Then $B_{0\perp}^\perp \supset B_0$ and equality holds if and only if B_0 is closed and linear in B. If $B_\perp'^\perp = B'$, then B' is closed and linear but the converse need not hold unless B is reflexive. It is known that if B_0 is a closed linear subspace of B, then $B_0^\perp$ is equivalent (that is, linearly isometric) to $(B/B_0)^*$ and $B^*/B_0^\perp$ is equivalent to B_0^* under the natural mappings.

We begin with a preliminary lemma which shows the existence of a considerable amount of freedom in the choice of definition of the special modulus of convexity of a space.

5.1 LEMMA.

$$\inf_{\|b_1 - b_2\| = \epsilon, \|b_i\| = 1} (2 - \| b_1 + b_2 \|) = \inf_{\|b_1 - b_2\| \geqq \epsilon, \|b_i\| \leqq 1} (2 - \| b_1 + b_2 \|).$$

Obviously the second term is not greater than the first. On the other hand suppose that $\| b_i \| \leqq 1$ and $\| b_1 - b_2 \| = \xi \geqq \epsilon$. If $\epsilon = 2$, b_1 and b_2 are both already of norm 1 and the equality of the lemma is obvious. If $\epsilon < 2$ and $b_1 = -b_2$, then any chord of length ϵ has a midpoint of greater norm than 0; if $b_1 \neq -b_2$, by adding a suitable multiple of $b_1 + b_2$ two new points, b_3 and b_4, can be obtained for which $\| b_3 + b_4 \| \geqq \| b_1 + b_2 \|$, $\| b_3 - b_4 \| = \| b_1 - b_2 \|$ and $\| b_i \| = 1$ for $i = 3$ or 4, $\| b_j \| \leqq 1$, $j = 3, 4$. Then there is at least one chord $b_5 b_6$ of the unit sphere bisected at $(b_3 + b_4)/2$; by 2.1 $\| b_5 - b_6 \| \geqq \| b_3 - b_4 \| \geqq \epsilon$. Lemma 2.4 gives the existence of a pair of points b_7 and b_8 such that $(b_7 + b_8)/\| b_7 + b_8 \| = (b_5 + b_6)/\| b_5 + b_6 \|$ and $\| b_7 - b_8 \| = \epsilon$.

Note also that the proof of 2.4 shows that b_7 and b_8 are between b_5 and b_6 so $\| b_7 + b_8 \| \geqq \| b_5 + b_6 \| \geqq \| b_1 + b_2 \|$. Hence the left hand side of the desired equation is not greater than the right and equality follows.

Note that for uniformly convex spaces this is a considerable improvement over Lemma 1 of [4]. Also if $\delta(\epsilon)$ is defined by either side of the equation of the lemma, it is possible to show that δ is continuous on the right for every $\epsilon < 2$. In a finite dimensional space it can also be shown that this δ is continuous on the left; the argument uses compactness of the unit sphere and does not generalize. However, I know no example of a space for which the special modulus of convexity is discontinuous.

5.2 LEMMA. *If $b_1, \cdots, b_n$ are any points of a normed vector space B, there is a factor space $A = B/B_0$ such that (1) A is of dimension not greater than n and (2) $\| a_{b_i} \| = \| b_i \|$ for $i = 1, \cdots, n$.*

Let $\beta_1, \cdots, \beta_n$ be elements of B^* such that $\| \beta_i \| = 1$ and $\beta_i(b_i) = \| b_i \|$, $i = 1, \cdots, n$; let $B_0 = \{b \mid \beta_i(b) = 0 \text{ for } i = 1, \cdots, n\}$. Then if β is not linearly dependent on $\beta_1, \cdots, \beta_n$, Lemma 1 of [5] can be applied to show that a solution exists for the set of equations $\beta_1(b) = \beta_2(b) = \cdots = \beta_n(b) = 0$, $\beta(b) = 1$; that is, b is in B_0 but $\beta(b) \neq 0$ so $\beta \notin B_0^\perp$. Hence $B_0^\perp$ is the set of linear combinations of the β_i so $B_0^\perp$, and hence B/B_0, are of dimension $\leqq n$.

On the other hand, if $b \in a_{b_i}$, then $\| b_i \| = \beta_i(b_i) = \beta_i(b) \leqq \| b \|$ so $\| a_{b_i} \| = \| b_i \|$.

5.3 LEMMA. *If $A = B/B_0$ and B is uniformly convex with a modulus of convexity δ continuous on the left, then δ is a modulus of convexity for A.*

If $\| a_i \| = 1$ and $\| a_1 - a_2 \| \geqq \epsilon$, then for each $\mu > 0$ there exists a point b_i in a_i such that $\| b_i \| = \| a_i \| (1 + \mu)$. Then $\| b_1 - b_2 \| \geqq \| a_1 - a_2 \| \geqq \epsilon$ so $\| a_1 + a_2 \| \leqq \| b_1 + b_2 \| \leqq 2[1 - \delta(\epsilon/(1 + \mu))] (1 + \mu)$. By continuity on the left of δ, $\| a_1 + a_2 \| \leqq 2(1 - \delta(\epsilon))$.

5.4 LEMMA. *If δ is monotone, let $\delta_1(\epsilon) = \lim_{\epsilon' \to \epsilon -} \delta(\epsilon')$; then δ_1 is a modulus of convexity for B if and only if it is a modulus of convexity for every factor space of B, and if and only if δ_1 is a modulus of convexity for every two dimensional factor space of B.*

δ_1 is continuous on the left so, by 5.3, δ_1 is a modulus for every factor space if it is a modulus for B. If δ_1 is not a modulus for B, then there exist two points b_1 and b_2 of norm one such that $\| b_1 - b_2 \| \geqq \epsilon$ while $\| b_1 + b_2 \| > 2(1 - \delta_1(\epsilon))$. By 5.2 there is a two dimensional factor space A in which $\| a_{b_1} + a_{b_2} \| = \| b_1 + b_2 \| > 2(1 - \delta_1(\epsilon))$, $\| a_{b_1} - a_{b_2} \| = \| b_1 - b_2 \| \geqq \epsilon$ and $\| a_{b_i} \| \leqq \| b_i \| = 1$; by 5.1 applied to A, δ_1 is not a modulus of convexity for A.

Less precisely this gives

5.5 THEOREM. *B is uniformly convex if and only if there is a common modulus of convexity for the factor spaces of B, and if and only if there is a common modulus of convexity for the two dimensional factor spaces of B.*

6. Conjugate spaces.

The results of Part I and §5 can now be combined to yield the principal results of this paper.

6.1 LEMMA. *η is a modulus of flattening in B if and only if it is a modulus of flattening for every two dimensional subspace of B.*

Obviously $\sup_{\|b_i\|=1, \|b_1-b_2\|\leqq\epsilon} [2 - \| b_1 + b_2 \|]$ is not increased if the b_i are restricted to lie in a subspace B_0 of B, so the modulus of flattening of B is a modulus of flattening in any subspace. On the other hand, if η is not a modulus of flattening for B, there exist $\epsilon > 0$ and two points b_1 and b_2 such that $\| b_i \| = 1$, $\| b_1 + b_2 \| \leqq \epsilon$ and $[2 - \| b_1 + b_2 \|]/\| b_1 - b_2 \| > \eta(\epsilon)$. Since b_1 and b_2 determine a two dimensional subspace of B containing both $b_1 + b_2$ and $b_1 - b_2$, the same relations hold in that subspace and η is not a modulus of flattening there.

An analogue of 5.4 and 5.5 is given later [Theorem 6.10]; it can not easily be proved directly as far as I know.

6.2 LEMMA. *If B^* is uniformly convex, then B is uniformly flattened.*

Let B_0 be any two dimensional subspace of B and, as before, let $B_0^\perp = \{\beta \mid \beta(b) = 0 \text{ if } b \in B_0\}$; then $B^*/B_0^\perp$ is a two dimensional factor space of B^* and δ_1^*, as in 5.4, is a modulus of convexity for $B^*/B_0^\perp$. Since B_0^* is equivalent to $B^*/B_0^\perp$, by 4.3 B_0 has a modulus of flattening η determined only by the modulus of convexity δ^* in B^*. By 6.1 η is a modulus of flattening for B.

6.3 COROLLARY. *If B is uniformly convex, then B^* is uniformly flattened.*

If B_1 is the completion of B, then B_1 is known [6] to be reflexive so B_1 is equivalent to $(B^*)^*$. Since B is dense in B_1, B_1 is also uniformly convex; by 6.2 B^* is uniformly flattened.

A direct proof along the line of 6.2 also can be given without using reflexivity of B_1.

6.4 LEMMA. *If B^* is uniformly flattened, B is uniformly convex.*

Let A be any two dimensional factor space B/B_0; then $B_0^\perp$ is equivalent to $(B/B_0)^*$ and is therefore a two dimensional subspace of B^*. By 6.1 and 3.6, A has a modulus of convexity δ determined only by the modulus of flattening in B^*. By 5.5, B is uniformly convex.

To prove the corresponding result with B and B^* interchanged requires a lemma related to 5.4 but stronger for its space. A factor space B^*/B' will be called a *lower factor space* of B^* if there exists a subspace B_0 of B such that $B' = B_0^\perp$.

6.5 LEMMA. *If δ' is a common modulus of convexity for all the two dimensional lower factor spaces of B^*, then δ^* defined by $\delta^*(\epsilon) = \delta'(\epsilon/2)/2$ is a modulus of convexity for B^*.*

If $\| \beta_i \| = 1$ and $\| \beta_1 - \beta_2 \| \geqq \epsilon > 0$, there exist b_1 and b_2 of norm one in B such that $(\beta_1 - \beta_2)(b_1) \geqq \| \beta_1 - \beta_2 \| - \epsilon/2 = \epsilon/2$ and $(\beta_1 + \beta_2)(b_2) \geqq \| \beta_1 + \beta_2 \| - \delta'(\epsilon/2)$. If B_0 is a two dimensional subspace of B containing b_1 and b_2, $C = B^*/B_0^\perp$ is a two dimensional lower factor space of B^*. If $c(\beta)$ is the element of C in which β lies, $\| c(\beta_1) + c(\beta_2) \| = \inf_{\beta \in c(\beta_1+\beta_2)} \| \beta \|$, but $\| \beta \| \geqq \beta(b_2) = (\beta_1 + \beta_2)(b_2) \geqq \| \beta_1 + \beta_2 \| - \delta'(\epsilon/2)$ so $\| c(\beta_1) + c(\beta_2) \| \geqq \| \beta_1 + \beta_2 \| - \delta'(\epsilon/2)$. Similarly $\| c(\beta_1) - c(\beta_2) \| \geqq (\beta_1 - \beta_2)(b_1) \geqq \epsilon/2$. Applying the hypothesis to C, $2(1 - \delta'(\epsilon/2)) \geqq \| c(\beta_1) + c(\beta_2) \| \geqq \| \beta_1 + \beta_2 \| - \delta'(\epsilon/2)$, so $\| \beta_1 + \beta_2 \| \leqq 2(1 - \delta'(\epsilon/2)) +$

$\delta'(\epsilon/2) = 2(1 - \delta'(\epsilon/2)/2)$; hence the given function δ^* is a modulus of convexity in B^*.

6.6 LEMMA. *If B is uniformly flattened, B^* is uniformly convex.*

This is a consequence of 6.1, 3.6, and 6.5.

These lemmas can now be collected into one principal theorem.

6.7 THEOREM. *B is uniformly convex (flattened) if and only if B^* is uniformly flattened (convex) and a modulus in one space is determined by that in the other.*

6.8 COROLLARY. *Every complete uniformly flattened space is reflexive.*

By 6.7, B^* is uniformly convex, hence [6] B^* is reflexive, so B is reflexive.

6.9 LEMMA. *If B' is a finite dimensional subspace of B^*, and $B_0 = B'_\perp = \{b \mid \beta(b) = 0$ if $\beta \in B'\}$, then $B' = B_0^\perp = B'^\perp_\perp$.*

Take $\beta_1, \cdots, \beta_n$ linearly independent and such that their linear combinations fill up B'; then if β is not in B', the set of equations $\beta_1(b) = \cdots = \beta_n(b) = 0$, $\beta(b) = 1$ has a solution [5, Lemma 1] so β is not in $B_0^\perp$.

6.10 THEOREM. *B is uniformly flattened if and only if there is a common modulus of flattening for all the factor spaces of B and a common modulus of flattening for all the two dimensional factor spaces of B.*

If B is uniformly flattened, B^* is uniformly convex. If B/B_0 is any factor space of B, $B_0^\perp$ is equivalent to $(B/B_0)^*$ and has the modulus of convexity δ^* of B^*, hence B/B_0 has a modulus of flattening determined only by δ^*, so the factor spaces of B have a common modulus of flattening.

If B' is any two dimensional subspace of B^* and if $B_0 = B'_\perp$, by 6.9 $B' = B_0^\perp$ so B' is equivalent to the $(B/B_0)^*$. Therefore $(B/B_0)^*$ is two dimensional and, by 3.6, a modulus of convexity in B' is determined by the given common modulus of flattening of the two dimensional factor spaces of B. Hence the two dimensional subspaces of B^* have a common modulus of convexity, so B^* is uniformly convex and B is uniformly flattened.

7. Mapping constants.

In [2] and [4] I considered the norms of isomorphisms of certain finite dimensional spaces into a uniformly convex space B, Theorems 5.5 and 6.7 can be applied to give certain related results. Let A and B be normed vector spaces; then as in [4] let $k(A, B) = \sup_{||T|| \leqq 1} \inf_{||a||=1} ||Ta||$. Then $k(A, B) > 0$ if and only if A is isomorphic with some subspace of B. If L_n is the n-dimensional space of points $(t_1, \cdots, t_n)$ with $||(t_1, \cdots, t_n)|| = \sum_{i \leqq n} |t_i|$ and $M_n = L_n^*$ and if B is a space with a modulus of convexity δ which is continuous on the left, 5.1 and Lemma 4 of [4] assert that $k(L_{2n}, B) \leqq [1 - \delta(2k(L_{2n}, B))]^n$ and $k(M_n, B) \leqq [1 - \delta(2k(M_n, B))]^{n-1}$. These inequalities were used in [4] to prove that if B is uniformly convex, then $\lim_n k(L_n, B) = \lim_n k(M_n, B) = 0$. 7.2 is related to this result.

7.1 LEMMA. *For any A and B and any $\xi > 0$, there exists a factor space C of B^* such that $k(A^*, C) > k(A, B) - \xi$.*

If $k(A, B) = 0$, this is obvious; if not let T be any isomorphism of A into B of norm $\leqq 1$ and define T^* from B^* to A^* by $T^*\beta(a) = \beta(Ta)$ for all a in A,

β in B^*. It is well known that every α in A^* is the image under T^* of a β in B^* and that $T^*\beta = \theta$ means $T^*\beta(a) = 0$ for all a in A or $\beta(b) = 0$ for every b in B_0, the image of A under T; that is, T^* carries $B_0^\perp$ into the zero element of A. Hence T^{*-1} defines a 1-1 mapping T_1 of A^* onto $C = B^*/B_0$, where $T_1\alpha = c(\beta)$ if $T^*\beta = \alpha$ (as before, $c(\beta)$ is the element of C in which β lies).

$\| T \| \leqq 1$ so $\| T^* \| \leqq 1$; therefore $\| T_1\alpha \| = \inf_{\beta \epsilon T_1\alpha} \| \beta \| = \inf_{T^*\beta = \alpha} \| \beta \| \geqq \| \alpha \|$. By the equivalence of C and B_0^*, define $T_1\alpha(b)$ for b in B_0 by $T_1\alpha(b) = \beta(b)$ for any β in $T_1\alpha$. Then $\| T_1\alpha \| = \sup_{\| b \| \leqq 1, b \epsilon B_0} | T_1\alpha(b) |$; if $k \| a \| \leqq \| Ta \| \leqq \| a \|$ for all a, then $\| \alpha \| = \sup_{\| a \| \leqq 1} | \alpha(a) | = \sup_{\| a \| \leqq 1} | T_1\alpha(Ta) | \geqq \sup_{\| b \| \leqq k, b \epsilon B_0} | T_1\alpha(b) | = k \| T_1\alpha \|$ so $\| T_1 \| \leqq 1/k$. Hence $T' = T_1/\| T_1 \|$ is a mapping of A^* onto C such that $\| \alpha \| \geqq \| T'\alpha \| \geqq k \| \alpha \|$ for every α Hence $k(A^*, C) > k(A, B) - \xi$ if $k > k(A, B) - \xi$.

7.2 THEOREM. *If B is uniformly flattened, that is, if B^* is uniformly convex, then* $\lim_n k(L_n, B) = \lim_n k(M_n, B) = 0$.

If $A = L_n$, $C = C_n$ in 7.1, by 7.1, 5.5, and the result quoted from [4] $k(M_n, C_n) \leqq [1 - \delta(2k(M_n, C_n))]^{n-1}$; as n increases, just as in [4] $k(M_n, C_n) \to 0$. For any $\xi > 0$ it follows that $\lim_n k(L_n, B) - \xi \leqq 0$ so $\lim_n k(L_n, B) = 0$. Interchanging L_n and M_n in this argument gives $\lim_n k(M_n, B) = 0$.

7.3 COROLLARY. *If B or B^* is isomorphic to a uniformly convex space, then* $\lim_n k(L_n, B) = \lim_n k(M_n, B) = 0$.

This follows from 7.2 and the results of [4].

This corollary gives a necessary condition for isomorphism of B or B^* with a uniformly convex space. It is not known whether the condition is sufficient, or, even stronger, whether the condition $\lim_n k(L_n, B) = 0$ implies that B is isomorphic to a uniformly convex space. If the latter hypotheses were true, it would imply, by 7.3, the pretty result that every uniformly flattened space is isomorphic to a uniformly convex space; that is, that B is isomorphic to a uniformly convex space if and only if B^* is isomorphic to a uniformly convex space. This seems unlikely, but I know of no counterexamples.

UNIVERSITY OF ILLINOIS

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TRANSFORMATIONS OF WIENER INTEGRALS UNDER TRANSLATIONS

BY R. H. CAMERON AND W. T. MARTIN

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Introduction

In his paper on Generalized Harmonic Analysis (in which references to his earlier work are given) N. Wiener [I] defines an average or integral over the space C of all functions $x(t)$ continuous in $0 \leqq t \leqq 1$ and vanishing at $t = 0$. The integral proved to be very useful in the theory developed there. More recently, the present authors [II] have shown that the solution of a fairly general non-linear equation can be expressed in terms of certain Wiener integrals. In view of these rather distinct problems in which the Wiener integrals have been useful, it seems worthwhile to develop further certain aspects of the Wiener integral. In the present paper we show how the integral transforms under a translation, and we consider special cases of translations which seem to lead to rather interesting results. Our first (restricted) transformation theorem is as follows:

THEOREM 1. *Let $F[y]$ be a functional defined and Wiener summable over C, let $F[y]$ be bounded in $y(\cdot)$ for $y(\cdot)$ in any uniformly-bounded set, and let $F[y]$ be continuous in the sense that if $\{y^{(n)}(t)\}$ is any sequence of functions of C which converge uniformly in $0 \leqq t \leqq 1$ to $y(t)$ then*

$$\lim_{n\to\infty} F[y^{(n)}] = F[y]. \tag{1.1}$$

Let $x_0(t)$ be a given function of C with a first derivative $x_0'(t)$ of bounded variation in $0 \leqq t \leqq 1$. Then under the translation

$$y(t) = x(t) + x_0(t) \tag{1.2}$$

the Wiener integral undergoes the transformation

$$\int_C^W F[y]\, d_w y = e^{-\int_0^1 [x_0'(t)]^2 dt} \int_C^W F[x + x_0] e^{-2\int_0^1 x_0'(t)\, dx(t)}\, d_w x. \tag{1.3}$$

We will recall a basic property of the Wiener integral. Let n be a positive integer and let $H(\eta_1, \cdots, \eta_n)$ be a function of n numerical variables $\eta_1, \cdots, \eta_n$. Consider a functional $H[y(t_1), \cdots, y(t_n)]$ for $t_1, \cdots, t_n$ fixed. If H is Wiener summable and if $0 \leqq t_1 \leqq t_2 \leqq \cdots \leqq t_n \leqq 1$, then by the theory, the Wiener integral of H is given by

$$\begin{aligned}\int_C^W H[y(t_1), \cdots, y(t_n)]\, d_w y \\ = \frac{1}{[\pi^n t_1(t_2 - t_1) \cdots (t_n - t_{n-1})]^{1/2}} \int_{-\infty}^{\infty} d\eta_1 \cdots \int_{-\infty}^{\infty} d\eta_n \Big\{ H(\eta_1, \cdots, \eta_n) \\ \cdot \exp\left[-\frac{\eta_1^2}{t_1} - \frac{(\eta_2 - \eta_1)^2}{t_2 - t_1} - \cdots - \frac{(\eta_n - \eta_{n-1})^2}{t_n - t_{n-1}} \right] \Big\}.\end{aligned} \tag{1.4}$$

The idea to be used in the proof of Theorem 1 consists in approximating to the functional $F[y]$ by some functional which depends only upon the values of $y(t)$ at some set of n points. This will enable us to use the relation (1.4) and thus to work with the known transformation formulas of Lebesque integrals.

Our second (and more general) theorem is obtained by approximating the characteristic functional of a quasi-interval by continuous functionals, thus obtaining the measure of a translated quasi-interval. From limits of sums of quasi-intervals we find the measure of general translated sets and from it the translation formula for unrestricted Wiener summable functionals.

THEOREM 2. *Let $x_0(t)$ be a given continuous function vanishing at $t = 0$ and having a derivative $x_0'(t)$ of bounded variation $0 \leqq t \leqq 1$; let Γ be a Wiener measurable subset of C, and let $T\Gamma$ be the translation of Γ by $x_0(t)$; i.e., let $T\Gamma$ consist of all functions $x(\cdot)$ of C for which $x(t) \equiv y(t) - x_0(t)$ and $y(\cdot) \in \Gamma$. Then*

$$\text{meas}_w[\Gamma] = e^{-\int_0^1 [x_0'(t)]^2 dt} \int_{T\Gamma}^{W} e^{-2\int_0^1 x_0'(t)\,dx(t)}\, d_w x. \tag{1.5}$$

Moreover if $F[y]$ is any functional for which either member of (1.6) *(below) exists, then the other member also exists and the equation* (1.6) *holds:*

$$\int_{\Gamma}^{W} F[y]\, d_w y = e^{-\int_0^1 [x_0'(t)]^2 dt} \int_{T\Gamma} F[x + x_0] e^{-2\int_0^1 x_0'(t)\,dx(t)}\, d_w x. \tag{1.6}$$

2. Proof of Theorem 1

Let n be a positive integer, denote by t_j the point

$$t_j = \frac{j}{n} \tag{2.1}$$

and for any function $y(t)$ of C define a polygonalized form of $y(t)$ by the relations

$$y_n(t) = y(t_j) + \frac{y(t_{j+1}) - y(t_j)}{t_{j+1} - t_j}\,(t - t_j); \qquad \text{for } \begin{matrix} t_j \leqq t \leqq t_{j+1}, \\ j = 0, 1, \cdots, n-1. \end{matrix} \tag{2.2}$$

The subscript n on a function $y(t)$, $x(t)$, $x_0(t)$ etc., will denote that this polygonalizing process has been carried out on the function.

We form our functional $F[y_n]$ for the functions of (2.2). Since $y_n(t)$ depends only upon the values of $y(t)$ at the n points $t_1, \cdots, t_n$, it follows that there exists a function

$$H(\eta_1, \cdots, \eta_n) \tag{2.3}$$

of n real variables $\eta_1, \cdots, \eta_n$ such that

$$F[y_n] = H[y(t_1), \cdots, y(t_n)]. \tag{2.4}$$

Next let M be any positive number. We define two subsets C_M and $C_{M,n}$ of C. The set C_M consists of all functions $y(\cdot)$ of C for which $|\, y(t)\,| \leqq M$ for

all t in $0 \leqq t \leqq 1$, and $C_{M,n}$ consists of all functions $y(\cdot)$ of C for which $|\, y(j/n)\,| \leqq M$ for $j = 1, \cdots, n$. We note that

$$C_{M,n} \supset C_{M,2n}\,, \qquad \lim_{n\to\infty} C_{M,2^n} = C_M\,,$$
$$C_M \subset C_{M'} \quad \text{if } M < M', \qquad \lim_{M\to\infty} C_M = C. \tag{2.5}$$

We will denote by TC_M and $TC_{M,n}$ the images of C_M and $C_{M,n}$ under the translation (1.2). For example, TC_M consists of all functions $x(\cdot)$ of C for which $x(t) = y(t) - x_0(t)$ and $y(\cdot) \in C_M$.

Now by (1.4) and (2.4) we have

$$\int_{C_{M,n}}^{W} F[y_n]\, d_w y = \gamma_n \int_{-M}^{M} \cdots \int_{-M}^{M} H(\eta_1, \cdots, \eta_n) e^{-\frac{\eta_1^2}{t_1} - \sum_{j=2}^{n} \frac{(\eta_j - \eta_{j-1})^2}{t_j - t_{j-1}}}\, d\eta_1 \cdots d\eta_n\,, \tag{2.6}$$

where

$$\gamma_n = \pi^{-n/2}[t_1(t_2 - t_1) \cdots (t_n - t_{n-1})]^{-\frac{1}{2}}.$$

We next observe that if $y(t)$ is the image of $x(t)$ under the translation (1.2) and if $x_n(t)$, $x_{0,n}(t)$ and $y_n(t)$ are the polygonalized functions corresponding to $x(t)$, $x_0(t)$ and $y(t)$ respectively, then

$$y_n(t) = x_n(t) + x_{0,n}(t). \tag{2.7}$$

If we write

$$\eta_j = y(t_j), \qquad \xi_j = x(t_j), \qquad a_j = x_0(t_j), \qquad j = 0, 1, \cdots, n, \tag{2.8}$$

then under (2.7) the ξ's undergo the translation

$$\eta_j = \xi_j + a_j\,, \qquad j = 1, \cdots, n. \tag{2.9}$$

On applying the transformation (2.9) to the n-fold Lebesgue integral in (2.5) we find that

$$\int_{C_{M,n}}^{W} F[y_n]\, d_w y = \gamma_n \int_{-M-a_1}^{M-a_1} d\xi_1 \cdots \int_{-M-a_n}^{M-a_n} d\xi_n \Big[H(\xi_1 + a_1, \cdots, \xi_n + a_n) \cdot e^{-\sum_{j=1}^{n} \frac{(\xi_j - \xi_{j-1})^2}{t_j - t_{j-1}}} \cdot e^{-\sum_{j=1}^{n} \frac{(a_j - a_{j-1})^2}{t_j - t_{j-1}}} \cdot e^{-2\sum_{j=1}^{n} \frac{(a_j - a_{j-1})(\xi_j - \xi_{j-1})}{t_j - t_{j-1}}} \Big]. \tag{2.10}$$

The next step is to interpret this n-fold Lebesgue integral as a Wiener integral (see (1.4)). Interpreting it this way and using (2.8) and the fact that

$$x(t_j) = x_n(t_j), \qquad x_0(t_j) = x_{0,n}(t_j),$$

we find

$$\int_{C_{M,n}}^{W} F[y_n]\, d_w\, y = e^{-\sum_{j=1}^{n} \frac{[x_0(t_j)-x_0(t_{j-1})]^2}{t_j-t_{j-1}}} \int_{TC_{M,n}}^{W} F[x_n + x_{0,n}] \cdot e^{-2\sum_{j=1}^{n} [x_0(t_j)-x_0(t_{j-1})][x(t_j)-x(t_{j-1})]}\, d_w\, x. \tag{2.11}$$

This gives us a transformation formula over $C_{M,n}$ and hence over $C_{M,2^n}$ for the polygonalized functions under the translation (2.7). To obtain our desired transformation formula (1.3) we let n approach infinity, (over the sequence $1, 2, 2^2, 2^3, \cdots$), and then M. We will use the following two lemmas.

LEMMA 1. *If $y_n(t)$ is the polygonalized form of $y(t)$ as in* (2.2) *then*

$$\lim_{n\to\infty} \int_{C_{M,2^n}}^{W} F[y_{2^n}]\, d_w\, y = \int_{C_M}^{W} F[y]\, d_w\, y. \tag{2.12}$$

LEMMA 2.

$$\lim_{n\to\infty} \sum_{j=1}^{n} \left\{ \frac{[x_0(t) - x_0(t_{j-1})]^2}{t_j - t_{j-1}} + 2\, \frac{[x_0(t_j) - x_0(t_{j-1})]\,[x(t_j) - x(t_{j-1})]}{t_j - t_{j-1}} \right\} = \int_0^1 [x_0'(t)]^2\, dt + 2\int_0^1 x_0'(t)\, dx(t), \tag{2.13}$$

the convergence being bounded in $x(\cdot)$ for all $x(\cdot)$ in any uniformly bounded set.

REMARK: Since n now varies, it must be mentioned that the points $t_j = j/n$ vary with n. For the sake of simplicity in writing, we do not add another index n to the t_j.

Lemmas 1 and 2 when applied to (2.11) will yield the desired theorem at once.

3. Proof of Lemma 1

Let $y(t)$ be a given function of C and let ϵ be a positive number. Then there exists an integer $n_0 = n_0(\epsilon)$ such that

$$|y(t') - y(t'')| < \frac{\epsilon}{2} \quad \text{for all} \quad |t' - t''| \leqq \frac{1}{n_0}. \tag{3.1}$$

With $n \geqq n_0$ and $t_j \leqq t \leqq t_{j+1}$ we have by (2.2) and (3.1)

$$|y(t) - y_n(t)| \leqq \left| \frac{y(t_{j+1}) - y(t_j)}{t_{j+1} - t_j} \right| |t - t_j| + |y(t_j) - y(t)| \leqq n \cdot \frac{\epsilon}{2} \cdot \frac{1}{n} + \frac{\epsilon}{2} = \epsilon. \tag{3.2}$$

Hence

$$\lim_{n\to\infty} y_n(t) = y(t) \quad \text{uniformly in} \quad 0 \leqq t \leqq 1, \tag{3.3}$$

and by the continuity of F (see (1.1)) we have

$$\lim_{n\to\infty} F[y_n] = F[y]. \tag{3.4}$$

Also, since by hypothesis F is bounded over C_M the convergence in (3.4) is bounded for $y(\cdot)$ in C_M. Thus (2.12) holds for each fixed positive number M. This yields Lemma 1.

4. Proof of Lemma 2

For convenience in writing we define

$$\psi[x(\cdot); t] = x_0(t) + 2x(t) \tag{4.1}$$

and

$$P_n[x] = \sum_{j=1}^{n} \frac{x_0(t_j) - x_0(t_{j-1})}{t_j - t_{j-1}} \{\psi[x(\cdot); t_j] - \psi[x(\cdot); t_{j-1}]\}. \tag{4.2}$$

Then by the mean-value theorem

$$P_n[x] = \sum_{j=1}^{n} x_0'(t_j^*)\{\psi[x(\cdot); t_j] - \psi[x(\cdot); t_{j-1}]\} \tag{4.3}$$

for some t_j^* in $t_{j-1} \leqq t_j^* \leqq t_j$. By the definition of the Stieltjes integral, since $x_0'(t)$ is of bounded variation and $\psi[x(\cdot), t]$ is continuous in t, this expression $P_n[x]$ approaches the integral

$$\int_0^1 x_0'(t)\, d_t\psi[x(\cdot); t] \tag{4.4}$$

as $n \to \infty$. Thus, we have proved the convergence in (2.13). We have yet to show that the convergence is bounded in $x(\cdot)$ for all $x(\cdot)$ in any uniformly bounded set. This is quite easy. Let B be any uniformly bounded set of C. This means that there is a constant $K = K_B$ such that

$$|x(t)| \leqq K \quad \text{for} \quad x(\cdot) \in B. \tag{4.5}$$

We apply Abel's transformation to (4.3); it yields

$$P_n(x) = x_0'(t_n^*)\psi[x(\cdot); 1] - \sum_{j=1}^{n-1} \{x_0'(t_{j+1}^*) - x_0'(t_j^*)\}\psi[x(\cdot); t_j]. \tag{4.6}$$

Now $\psi[x(\cdot); t] \equiv x_0(t) + 2x(t)$ is bounded in t and $x(\cdot)$ for $0 \leqq t \leqq 1$, $x(\cdot) \in B$, and also $x_0'(t)$ is of bounded variation in $0 \leqq t \leqq 1$. Hence the $P_n[x]$ are bounded in n and $x(\cdot) \in B$. This yields Lemma 2.

On applying Lemmas 1 and 2 and the convergence property (1.1) of F to equation (2.11) we obtain

$$\int_{C_M}^{W} F[y]\, d_w y = \int_{TC_M}^{W} F[x + x_0] e^{-\int_0^1 [x_0'(t)]^2 dt - 2\int_0^1 x_0'(t)\, dx(t)}\, d_w x. \tag{4.7}$$

On letting M approach infinity we obtain the desired result (1.3).

5. Proof of Theorem 2

We begin our proof by eastblishing (1.5) and we first do this in the special case in which $\Gamma = I$, where I is the quasi-interval

$$\xi_j' \leqq y(t_j) \leqq \xi_j'' \qquad \left\{\begin{matrix} j = 1, \cdots, n \\ 0 < t_1 < \cdots < t_n \leqq 1 \end{matrix}\right\}. \tag{5.1}$$

(It is permissible for any ξ_j' to be $-\infty$ and any ξ_j'' to be $+\infty$.) Now let $\epsilon > 0$ and let $\varphi_{j,\epsilon}(\eta)$ be a continuous "trapezoidal" function which equals zero outside the interval $\xi_j' - \epsilon < \eta < \xi_j'' + \epsilon$, equals unity inside the interval $\xi_j' < \eta < \xi_j''$, and is linear on the remaining intervals. (If ξ_j' takes on the improper value $-\infty$, so does $\xi_j' - \epsilon$, etc.)

Let $\chi_I(y)$ be the characteristic functional of I, (i.e., $\chi_I(y) = 1$ if $y \in I$ and 0 otherwise), and let

$$\chi_{I,\epsilon}(y) = \prod_{j=1}^{n} \varphi_{j,\epsilon}(y(t_j)). \tag{5.2}$$

It is easy to see that $F(y) = \chi_{I,\epsilon}(y)$ satisfies the hypotheses of Theorem 1, for boundedness and Wiener summability are obvious, and continuity in the uniform topology follows from the continuity of the individual $\varphi_{j,\epsilon}(\eta)$. Hence (1.3) holds for $\chi_{I,\epsilon}$ and we have

$$\int_C^W \chi_{I,\epsilon}(y)\, d_w y = e^{-\int_0^1 [x_0'(t)]^2 dt} \int_C^W \chi_{I,\epsilon}[x + x_0] e^{-2\int_0^1 x_0'(t)dx(t)}\, d_w x. \tag{5.3}$$

But it is also obvious that for each y in C, $\lim_{\epsilon \to 0} \chi_{I,\epsilon}[y] = \chi_I[y]$, for if $y \in I$, $\chi_{I,\epsilon}[y] = 1$ for all ϵ, while if $y \notin I$, at least one of the inequalities (5.1) is not satisfied, and for sufficient small ϵ the corresponding $\varphi_{j,\epsilon}(y(t_j))$ will be zero. Thus letting $\epsilon \to 0$ and applying the principle of monotonic convergence to (5.3), we have

$$\int_C^W \chi_I(y)\, d_w y = e^{-\int_0^1 [x_0'(t)]^2 dt} \int_C \chi_I[x + x_0] e^{-2\int_0^1 x_0(t)dx(t)}\, d_w x,$$

which is equivalent to (1.5) with $\Gamma = I$.

We next establish (1.5) in general by progressively enlarging the scope of Γ. Since both sides of (1.5) are completely additive functions of Γ, it follows at once that (1.5) holds for any enumerable set of quasi-intervals which have only boundary points in common. But any measurable set Γ can be expressed as a set Γ_0 of measure zero plus the limit of a decreasing sequence of sets $\Gamma_1 \supset \Gamma_2 \supset \cdots$ in which each Γ_k is the sum of an enumerable set of quasi-intervals with only boundary points in common. Applying (1.5) to each Γ_k and taking the limit as $k \to \infty$, we obtain (1.5) for Γ by monotonic convergence. Thus (1.5) holds in general.

Coming now to a consideration of (1.6), we note that

$$E(x) \equiv e^{-\int_0^1 [x_0'(t)]^2 dt - 2\int_0^1 x_0(t)dx(t)} \tag{5.4}$$

is a Wiener measurable functional of $x(\cdot)$, and hence that the measurability of $F(y)$ on Γ implies the measurability of $E(x)F[x + x_0]$ on $T\Gamma$. We need therefore only consider the absolute value of the functional so far as integrability is concerned.

Let us first consider the case in which the functional F is bounded and non-negative. Let $0 \leqq F(y) < M$ on Γ, where M is an integer, and let us consider, for a fixed positive integer n, the sets Γ_k such that $y \in \Gamma_k$ if $y \in \Gamma$ and if

$$\frac{k-1}{n} \leqq F(y) < \frac{k}{n}, \qquad k = 1, 2, \cdots, Mn. \tag{5.5}$$

Then we have

$$\frac{k-1}{n} \operatorname{meas}_w (\Gamma_k) \leqq \int_{\Gamma_k}^{w} F(y)\, d_w y \leqq \frac{k}{n} \operatorname{meas}_w (\Gamma_k), \tag{5.6}$$

and from (1.5), (5.4), and (5.5) we have

$$\begin{aligned} \frac{k-1}{n} \operatorname{meas}_w (\Gamma_k) = \frac{k-1}{n} \int_{T\Gamma_k}^{w} E[x]\, d_w x \leqq \int_{T\Gamma_k} F[x + x_0]E[x]\, d_w x \\ \leqq \frac{k}{n} \int_{T\Gamma_k}^{w} E[x]\, d_w x = \frac{k}{n} \operatorname{meas}_w (\Gamma_k). \end{aligned} \tag{5.7}$$

But (5.6) and (5.7) imply that

$$\left| \int_{\Gamma_k}^{w} F[y]\, d_w y - \int_{T\Gamma_k}^{w} F[x + x_0]E(x)\, d_w x \right| \leqq \frac{1}{n} \operatorname{meas}_w (\Gamma_k)$$

and we obtain on adding for $k = 1, 2, \cdots, Mn$

$$\left| \int_{\Gamma} F[y]\, d_w y - \int_{T\Gamma} F[x + x_0]E(x)\, d_w x \right| \leqq \frac{1}{n} \operatorname{meas}_w (\Gamma).$$

Letting $n \to \infty$, we obtain the equality of these integrals, and (1.6) is established for non-negative bounded functionals.

Next take a non-negative functional F which is not bounded, and let $F_M(y) = \min (M, F(y))$. Then we have (1.6) holding for F_M and hence for F, by monotonic convergence. Finally, if F is any real functional, the theorem holds for $|F|$. Then if the integrals exist for $|F|$, they exist for the positive part of F and the negative part of F, and (1.6) holds for both and hence for F itself. This concludes the proof of Theorem 2.

6. An application of the translation theorem

By applying the translation formulas of Theorems 1 and 2 to various special cases of functionals $F[y]$ we are able to obtain formulas for the evaluation of various Wiener integrals. To illustrate this, we shall give a new proof of a special case of a theorem of Paley and Wiener [III].

LEMMA 3. *Let $\alpha(t)$ be real and of bounded variation on $0 \leqq t \leqq 1$, and let α be so normalized that*

$$\alpha(1) = 0. \tag{6.1}$$

Let

$$A = \sqrt{\int_0^1 [\alpha(t)]^2\, dt} \tag{6.2}$$

and let $G(u)$ be a (real or complex) measurable function defined on $-\infty < u < \infty$. Then a necessary and sufficient condition that

$$G\left[\int_0^1 x(t)\, d\alpha(t)\right]$$

be a Wiener measurable function of $x(\cdot)$ over C is that

$$e^{-u^2} G(Au)$$

be of class L_1 on $-\infty < u < \infty$. Moreover, if this condition is satisfied,

$$\int_C^W G\left[\int_0^1 x(t)\, d\alpha(t)\right] d_w x = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2} G(Au)\, du. \tag{6.3}$$

Before giving the proof of this lemma, we will derive a corollary to it.

COROLLARY. *Let $\beta(t)$ be a real function of class L_1 on $0 \leqq t \leqq 1$, let*

$$A = \sqrt{\int_0^1 \left[\int_t^1 \beta(\xi)\, d\xi\right]^2 dt}, \tag{6.4}$$

and let $G(u)$ be a (complex) measurable function on $-\infty < u < \infty$. Then a necessary and sufficient condition that

$$G\left[\int_0^1 \beta(t)x(t)\, dt\right]$$

be Wiener summable over C is that $e^{-u^2} G(Au)$ be of class L_1 on $-\infty < u < \infty$. Moreover, we will then have

$$\int_C^W G\left[\int_0^1 \beta(t)x(t)\, dt\right] d_w x = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2} G(Au)\, du. \tag{6.5}$$

The corollary follows from Lemma 3 by putting

$$\alpha(t) = -\int_t^1 \beta(\xi)\, d\xi. \tag{6.6}$$

By specializing the function G in (6.5) we obtain various results. For example, if $G(u) = u^{2n}$ we find

$$\begin{aligned} \int_C^W \left[\int_0^1 \beta(t)x(t)\, dt\right]^{2n} d_w x &= \frac{A^{2n}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2} u^{2n}\, du \\ &= \frac{A^{2n}}{\pi}\, \Gamma\left(\frac{2n+1}{2}\right). \end{aligned} \tag{6.7}$$

In the next section, we prove Lemma 3.

7. Proof of Lemma 3

We carry through the proof of Lemma 3 in successive stages.

STEP 1. We establish (6.3) for

$$G(u) = e^{\lambda u} \tag{7.1}$$

where λ is an arbitrary real number. Take

$$x_0(t) = \frac{\lambda}{2}\int_0^t \alpha(\xi)\, d\xi \tag{7.2}$$

and apply the translation formula (1.3) of Theorem 1 to the integral $\int_C^W 1 \cdot d_w y$ under the translation $y(t) = x(t) + x_0(t)$. Thus

$$1 = \int_C^W 1 \cdot d_w y = \int_C^W 1 \cdot e^{-\int_0^1 [x_0'(t)]^2 dt - 2\int_0^1 x_0'(t)\, dx(t)}\, d_w x, \tag{7.3}$$

so that

$$e^{\int_0^1 [x_0'(t)]^2 dt} = \int_C^W e^{-2\int_0^1 x_0'(t)\, dx(t)}\, d_w x, \tag{7.4}$$

or

$$\begin{aligned} e^{\frac{\lambda^2}{4}\int_0^1 [\alpha(t)]^2 dt} &= \int_C^W e^{-\lambda\int_0^1 \alpha(t)\, dx(t)}\, d_w x \\ &= \int_C^W e^{\lambda\int_0^1 x(t)\, d\alpha(t)}\, d_w x. \end{aligned} \tag{7.5}$$

A direct calculation yields

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-u^2} e^{\lambda A u}\, du = e^{\frac{\lambda^2 A^2}{4}} \tag{7.6}$$

and thus (6.3) holds for $G(u) = e^{\lambda u}$ with λ real:

$$\int_C^W e^{\lambda\int_0^1 x(t)\, d\alpha(t)}\, d_w x = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-u^2} e^{\lambda A u}\, du. \tag{7.7}$$

STEP 2. We next establish (6.3) for $G(u) = e^{\lambda u}$ with λ complex. (While the Wiener integral is originally defined for real functionals, we can easily extend the definition to complex functionals by breaking into real and imaginary parts).

If A is real, we have (6.3) for $G(u) = e^{Au}$ and e^{-Au} and hence for $G(u) = \cosh Au$; the existence of the integral is guaranteed by our translation theorem, as applied in Step 1. Now in the domain $|\lambda| < A$, the function $e^{\lambda u}$ is dominated by $2 \cosh Au$. If we consider the left member of (7.7) as a function of λ and

integrate around a contour, we are able to interchange the order of integration by the mixed Fubini theorem. But the integrand $e^{\lambda\int_0^1 x(t)d\alpha(t)}$ is analytic in λ for each $x(\cdot)$ in C, and its integral around a closed contour in the λ- plane is zero. Hence, by Morera's theorem the integral

$$\int_C^W e^{\lambda\int_0^1 x(t)d\alpha(t)}\, d_w x$$

is analytic in λ. Obviously, the right member of (7.7) is analytic in λ. Since both members of (7.7) are analytic in λ and since they are equal for λ real, the equality (7.7) must persist for complex values of λ. This completes Step 2.

STEP 3. We establish (6.3) when $G(u)$ is a real absolutely continuous function vanishing outside a finite interval and having an essentially bounded derivative.

Let $g(v)$ be the Fourier transform

$$g(v) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iuv}G(u)\, du. \tag{7.8}$$

Then $g(v)$ is an entire function of v and is of class L_1 on the real axis, and

$$G(u) = \int_{-\infty}^{\infty} e^{iuv}g(v)\, dv. \tag{7.9}$$

Thus

$$\begin{aligned}
\int_C^W G\left[\int_0^1 x(t)\, d\alpha(t)\right] d_w x &= \int_C^W \left\{\int_{-\infty}^{\infty} e^{iv\int_0^1 x(t)d\alpha(t)}\, g(v)\, dv\right\} d_w x \\
&= \int_{-\infty}^{\infty} g(v)\left\{\int_C^W e^{iv\int_0^1 x(t)d\alpha(t)}\, d_w x\right\} dv \\
&= \int_{-\infty}^{\infty} g(v)\left\{\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-t^2}e^{ivAt}\, dt\right\} dv \\
&= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-t^2}\left\{\int_{-\infty}^{\infty} e^{ivAt}g(v)\, dv\right\} dt \\
&= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-t^2}G(At)\, dt,
\end{aligned}$$

and (6.3) holds for the function G of Step 3. (The changes of integration are justified by Fubini's theorem, since $g(v)$ is of class L_1, and the complex exponentials are of absolute value 1).

STEP 4. We establish (6.3) when $G(u)$ is a real bounded measurable function vanishing outside a finite interval. This is done by the use of a smoothing process on $G(u)$ and the principle of dominated convergence. We omit the details.

STEP 5. We next establish the general theorem for real non-negative $G(u)$. This is established in the customary way by use of the principle of monotone

convergence. The existence of one side proves the existence of the other, and hence, we obtain the desired necessary and sufficient condition of Wiener summability.

STEP 6. The case of general complex-valued functions is established by combining the four parts, real positive, real negative, pure imaginary positive and pure imaginary negative.

This establishes the general lemma, including summability of the integral.

Lemma 3 and the Corollary furnish means of calculating various Wiener integrals.

THE MASSACHUSETTS INSTITUTE OF TECHNOLOGY
CAMBRIDGE, MASSACHUSETTS, AND
SYRACUSE UNIVERSITY, SYRACUSE, NEW YORK

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